Due: 30 March

Problem Set E
Problem Set*

Instructions: You are encouraged work with your classmates on this assignment. (Please take the time to think hard about these problems before you meet in groups.) However, each individual must write up their own solutions.

Honor code: You may use your class notes, refer to the book. However, no outside resources (including books, the internet, *etc.*).

Problem 1. Any line ℓ in the plane satisfies an equation of the form

$$Ax + By = C$$

where $A, B, C \in \mathbb{R}$ and A and B are not both zero.

i. Show that the above equation is equivalent to one of the form

$$\Re(z \cdot (\cos\theta + \imath \sin\theta)) = r$$

where $\theta \in \mathbb{R}$ and $r \geq 0$.

ii. Give a geometric interpretation to θ and r.

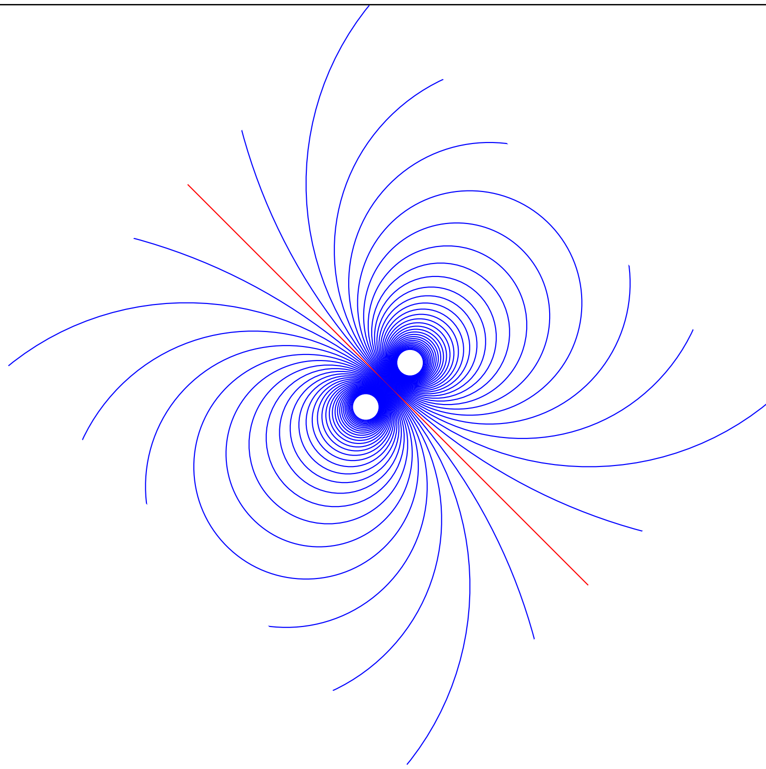
Problem 2 (10 pts.)**.** Let z_0 and z_1 be two distinct points in $\mathbb{C}$. Show that the locus of the equation

$$|z - z_0| = \rho|z - z_1|$$

is

- a line when $\rho = 1$, and
- a circle (of postive radius) when $0 < \rho < 1$ and when $\rho > 1$.

*All problems are worth five points, unless otherwise specified.



The *extended complex plane* $\widehat{\mathbb{C}}$ is the complex plane $\mathbb{C}$ together with a special point ∞:

$$\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}.$$

A line in $\widehat{\mathbb{C}}$ is a standard line in $\mathbb{C}$ together with ∞.

Problem 3. For any $z_0 \in \mathbb{C}$ define the *translation* $T_{z_0} : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$T_{z_0}(z) = \begin{cases} z + z_0 & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases}$$

The map T_{z_0} is a bijection. Prove that T_{z_0} takes lines to lines and circles to circles.

Problem 4. For any non-zero $\zeta \in \mathbb{C}$ define the *similarity* $\Lambda_\zeta : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$\Lambda_\zeta(z) = \begin{cases} \zeta \cdot z & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty \end{cases}$$

The map Λ_ζ is a bijection. Prove that Λ_ζ takes lines to lines and circles to circles.

Problem 5 (10 pts.)**.** Define inversion $I : \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ by

$$I(z) = \begin{cases} \frac{1}{z} & \text{if } z \in \mathbb{C}\backslash\{0\} \\ \infty & \text{if } z = 0 \\ 0 & \text{if } z = \infty \end{cases}$$

The function I is bijective on the extended complex plane.

i. Prove that I maps any line or circle passing through the origin to a line.

ii. Prove that I maps any line or circle not passing through origin to a circle.

Let a, b, c and d be complex numbers. Write

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

If $\det(\gamma) = ad - bc \neq 0$ and $c \neq 0$, define the *linear fractional transformation* associated with γ by

$$\ell_\gamma(z) = \begin{cases} \frac{az+b}{cz+d} & \text{if } cz + d \neq 0 \\ \frac{a}{c} & \text{if } z = \infty \\ \infty & \text{if } cz + d = 0. \end{cases}$$

In the case where $\det(\gamma) = ad - bc \neq 0$ and $c = 0$, define

$$\ell_\gamma(z) = \begin{cases} \frac{az+b}{d} & \text{if } z \in \mathbb{C} \\ \infty & \text{if } z = \infty. \end{cases}$$

Problem 6. Prove that translations, similarities and inversions are all of the form ℓ_γ for the appropriate matrix γ with non-zero determinant.

Problem 7. Prove that any linear fractional transformation ℓ_γ can be expressed as a composition of translations, similarities and inversions. Conclude from this that a linear fractional transformation takes lines and circles in $\widehat{\mathbb{C}}$ to lines or circles.