

Trigonometry in a Quick Turn

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Part 1

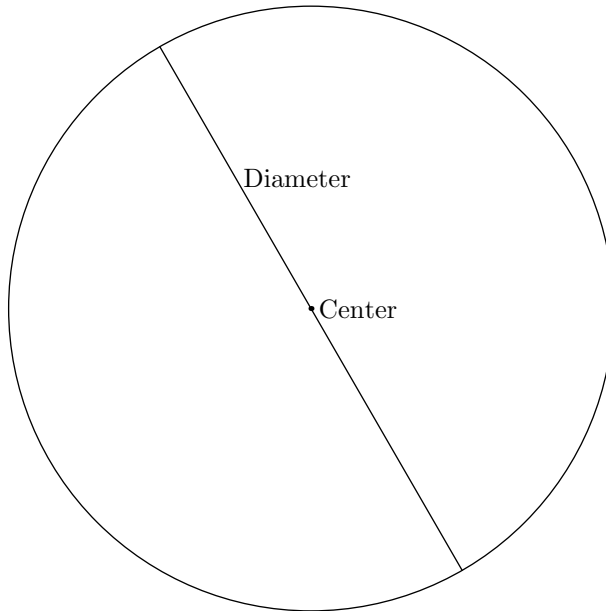
Geometry

CHAPTER 1

Circles

Our world, for the most part, will be the plane: an idealized flat surface extending indefinitely in every direction.

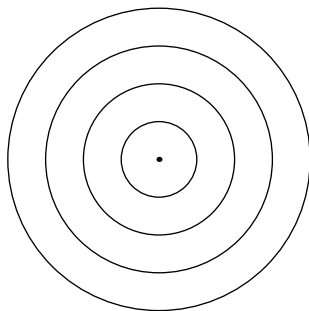
A *circle* is a set of points of a fixed distance from a given point. This distance is called the *radius* and the given point is called the *center*. The *diameter* is twice the radius.



Two pieces of information describe a circle: the diameter and the center. The *circumference* is the length around the circle or the *perimeter*. How does this quantity depend on the diameter and center?

If we change the center of a circle while keeping the diameter the same, does the circumference change? A little experimentation leads us to conclude not.

Does the circumference depend on the diameter? This time experimentation leads us to conclude: 'yes.' For example, if the diameter is doubled then the circumference is likewise doubled, If the diameter is tripled, the circumference is tripled, and so on.



Hence, the *ratio* of the circumference to the diameter is the same for all circles. This constant is labeled by the symbol: π . Hence

$$\frac{C}{d} = \pi$$

where d is the diameter and C is the circumference. Since $d = 2r$ (where r is the radius),

$$\frac{C}{2r} = \pi.$$

Multiplying both sides by $2r$, results in the equation for the circumference of the circle:

$$C = 2\pi r$$

Numerical computation of π has a long history. A passage in the Old Testament of the Bible suggests that it is three. This may well have been the usual value used in the ancient world to do various practical computations. However, a more accurate estimate can be found in the Rhind Papyrus. This document, a principal source of information about mathematics in ancient Egypt, was written by the scribe Ahmes around 1650 B.C. It is actually a copy of a document that originates two centuries earlier. At any rate, the estimate given there is

$$4 \cdot (1 - 1/9)^2 = 3.16\dots$$

More accurate estimates were given by the Greeks. The most sophisticated approach was due to Archimedes, who gave a procedure using inscribed and circumscribed polygons of the circle. This method was capable of producing estimated of arbitrary accuracy. As an example, Archimedes gives,

$$3 + \frac{10}{71} < \pi < 3 + \frac{1}{7}.$$

Various other estimates have been given since the time of Archimedes. Among these are Ptolemy (Egypt, 2nd century AD), Aryabhatta (India, 500 AD), and al-Kashi (Samarkand, 1424 AD) to mention only a few. By the time of the Enlightenment (1700's) the question of accuracy ceased to have practical importance (*i.e. in architecture, engineering,*

etc.). Indeed in 1719, M. de Lagny calculated the first 127 digits of the decimal expansion of π . The numerical computation of π remains of interest as a benchmark problem that measures how much is known in other areas of mathematics and computing. Modern computations of π , using computers together with advanced mathematical methods, have given its decimal expansion in the hundreds of billions of digits. The TI 85 calculator gives the estimate

$$\pi \approx 3.14159265$$

with an error of less than 10^{-8} .

Question 0.1. The circumference of the Earth around the equator has been measured to be 40,060 kilometers. What is the radius of the Earth, assuming that it is a perfect sphere?

Solution:

The circumference $C = 40,060$ kilometers. Now, $C = 2\pi r$ where r is the radius. So:

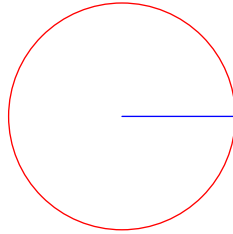
$$40060 = 2\pi r.$$

Solving for r yields

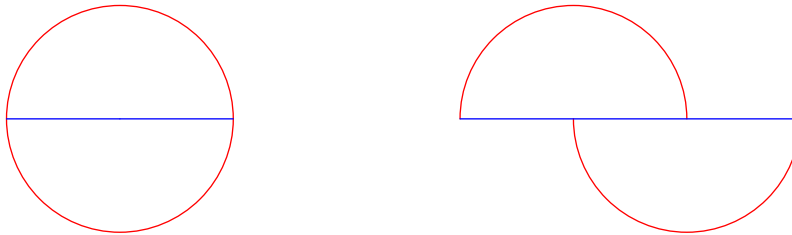
$$r = \frac{40060}{2\pi} = \frac{20030}{\pi} \approx 6375.7km.$$

CHAPTER 1. CIRCLES

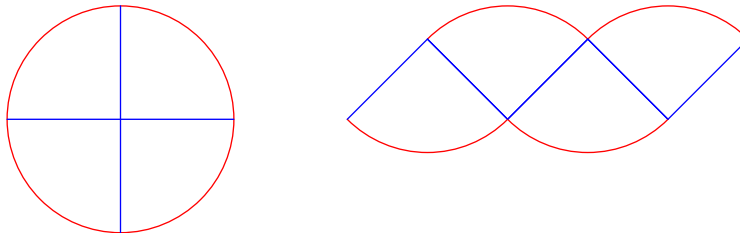
Since the radius of a circle determines the shape of the circle completely, it is reasonable to expect a formula for the area in terms of this radius. To deduce such a formula, let's start out with a circle of radius r .



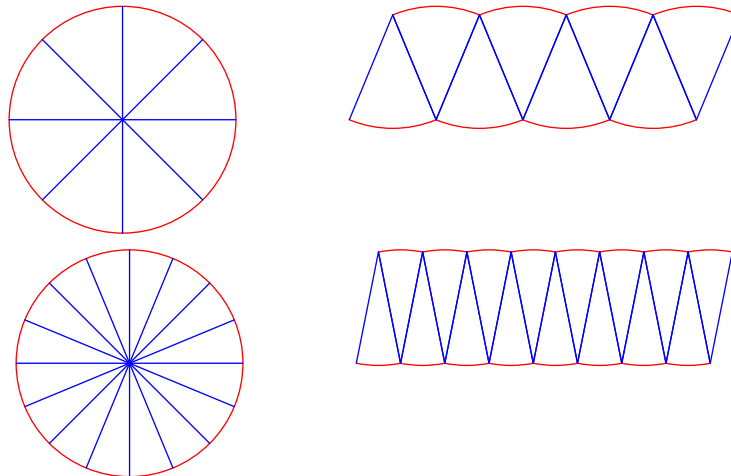
Dissect and re-arrange this circle as shown below.

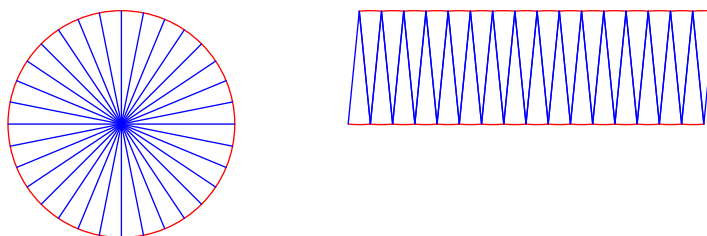


At the next stage, dissect into quarters and re-arrange:

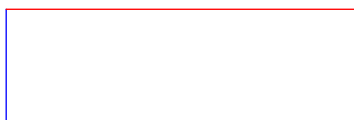


Now dissect into eighths and re-arrange, then into sixteenths and so on.





Observe that, as this process is repeated, the re-arranged shape resembles a rectangle.



The width of this rectangle is one-half the circumference of the original circle

$$\frac{1}{2}C = \frac{1}{2}2\pi r = \pi r$$

while the height is just the radius of the original circle: r . The area is

$$\text{width} \times \text{height} = (\pi r) \cdot r = \pi r^2.$$

This must also be the area within the original circle. Hence the formula for the area within a circle of radius r is

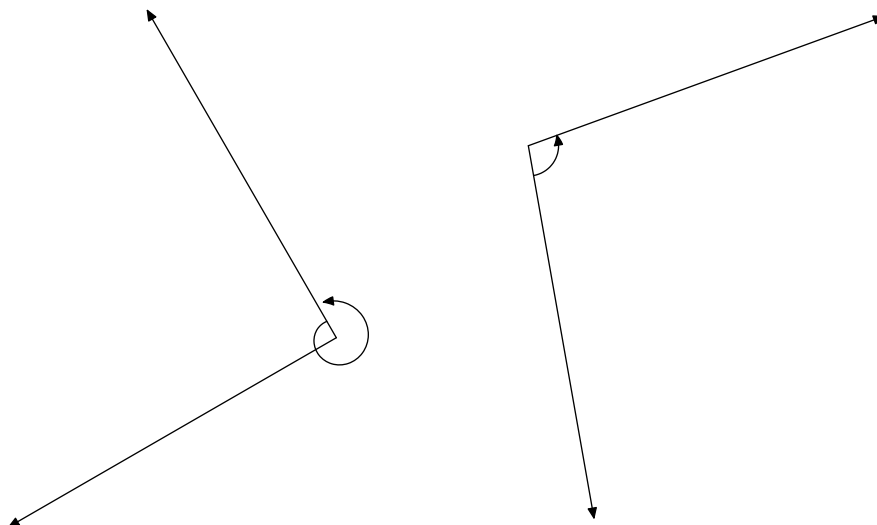
$$A = \pi r^2.$$

!

CHAPTER 2

Angles and Radian Measure

An *angle* consists of a point called the vertex together with two rays (half-lines) emanating from this vertex. An *oriented* angle is an angle together with a *rotation* carrying one ray, the *initial ray* to the other *terminal ray*.



Observe that an angle can have two possible orientations. If the rotation is counter-clockwise, it is called *positively oriented*. If the rotation is clockwise, it is *negatively oriented*.

Oriented angles can be measured. Draw a circle of unit radius with center at the vertex of the angle. The rotation of the initial ray to the terminal ray determines a path along this unit circle. The *radian measure* of the angle is

•

+ length of path

if the angle is positively oriented, and

•

– length of path

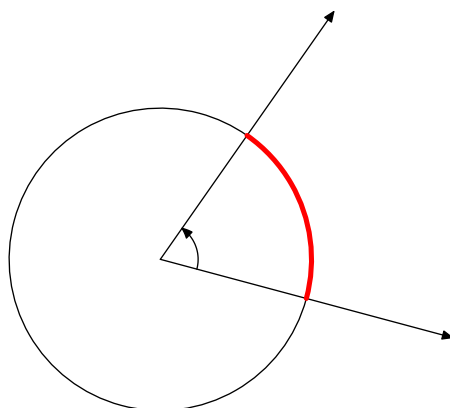
if the angle is negatively oriented.

CHAPTER 2. ANGLES AND RADIAN MEASURE

A little effort is required to understand this. Imagine a dog and its owner. The owner is standing at the vertex of the angle and is attached to the dog by a taut leash of unit length. The dog starts at the point where the initial ray meets the unit circle. It then walks along the unit circle so that the leash traces out the rotation of the angle. The dog will eventually stop at the point where the terminal ray meets the unit circle. The radian measure is

- the distance the dog travels, if the angle is positively oriented, and
- the negative of this distance, if the angle is negatively oriented.

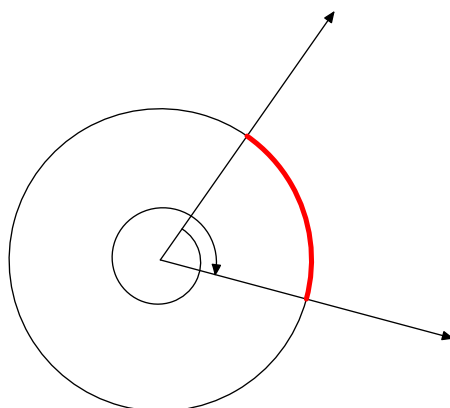
How does this apply to the angle shown below? (The circle has unit radius and center at the vertex of the angle.)



Since the orientation is counter-clockwise, the radian measure would be

length of the red arc.

How about the picture below? (The circle has unit radius and center at the vertex of the angle.)



CHAPTER 2. ANGLES AND RADIAN MEASURE

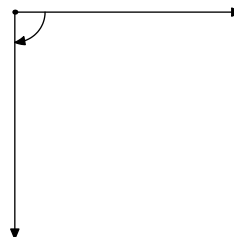
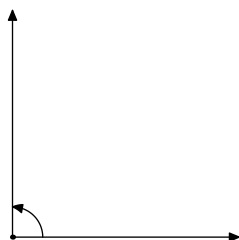
Here the orientation is clockwise, so the sign for the radian measure is $-$. The length of the path around the unit circle is

circumference of circle + length of the red arc.

The radian measure is then the negative of this number. Using work from the previous section, the circumference of the unit circle is 2π (the radius is one!). So the radian measure for this angle would be

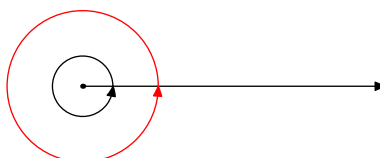
$-(2\pi + \text{length of the red arc})$.

Question 0.2. What are the radian measures of the following six oriented angles?

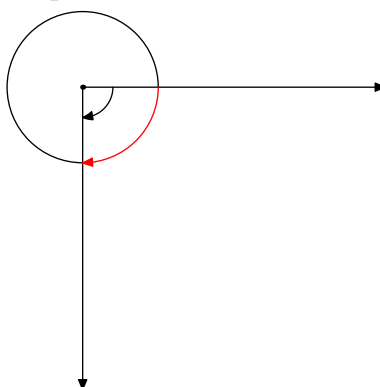


Solution:

In each figure, draw the unit circle centered at the vertex and consider the path determined by the angle. So, for example, the path for the first angle is one full circuit around in the counter-clockwise orientation:



Hence the radian measure of this angle is: 2π . Here is the picture for the last angle in the problem:



Here the path is a quarter circuit in the clockwise orientation (*i.e.* negative). So the radian measure is:

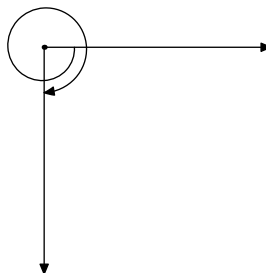
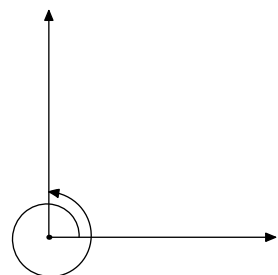
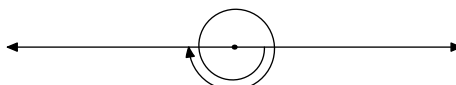
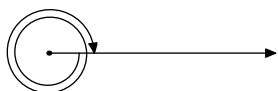
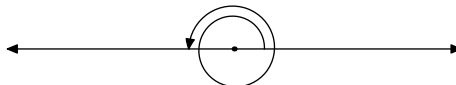
$$-\frac{2\pi}{4} = -\frac{\pi}{2}$$

The full list of answers is:

2π	π
-2π	$-\pi$
$\frac{\pi}{2}$	$-\frac{\pi}{2}$

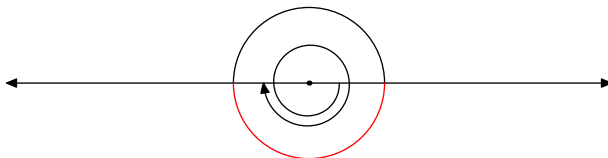
CHAPTER 2. ANGLES AND RADIAN MEASURE

Question 0.3. What are the radian measures of the following six oriented angles?



Solution:

Let's work through the fourth angle in the problem in detail. As before, draw the circle of radius one centered at the vertex of the angle.



The path along the circle, described by this angle, is one full circuit followed by a half circuit in the clockwise orientation. Hence the radian measure is:

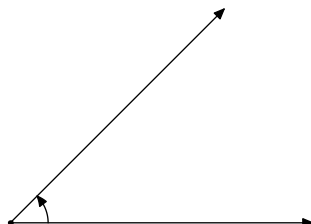
$$-(2\pi + \frac{1}{2}2\pi) = -3\pi.$$

All the angles in the problem can be understood in this way. The full table of answers is:

4π	3π
-4π	-3π
$\frac{5\pi}{2}$	$-\frac{5\pi}{2}$

CHAPTER 2. ANGLES AND RADIAN MEASURE

Question 0.4. The terminal ray in the diagram below is obtained by rotating the initial ray *one-eighth* of a full revolution in the counter-clockwise sense. What is the radian measure the angle?



CHAPTER 2. ANGLES AND RADIAN MEASURE

Solution:

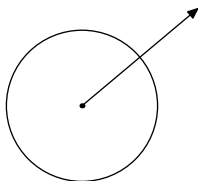
A full revolution in the counter-clockwise orientation is $+2\pi$ radians. One-eighth of a full revolution is therefore:

$$\frac{1}{8}2\pi = \frac{\pi}{4}.$$

CHAPTER 2. ANGLES AND RADIAN MEASURE

Question 0.5. Describe all possible oriented angles with the same initial and terminal rays? What are the radian measures of these angles?

Solution:



Certainly, one possible answer is 0 radians. But many alternatives exist: namely, any number full revolutions in the clockwise or counter-clockwise orientations:

$$\begin{aligned} &0 \\ &\pm 2\pi \\ &\pm 4\pi \\ &\pm 6\pi \\ &\vdots \end{aligned}$$

A succinct way of conveying this infinite list of possibilities is:

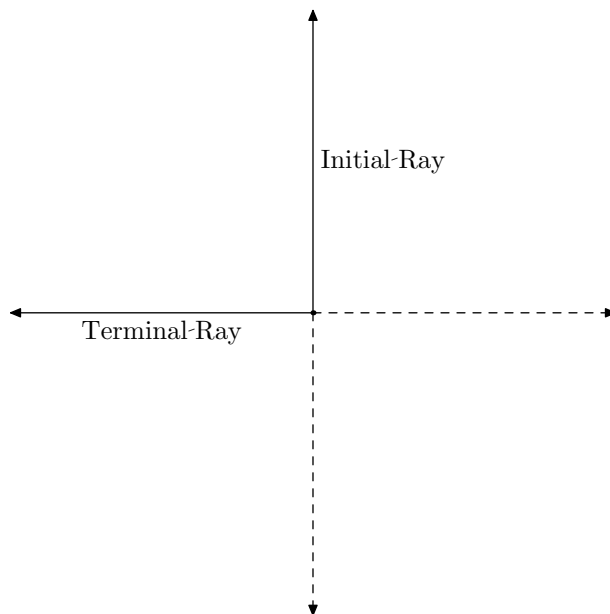
$$2\pi n \quad \text{where } n \text{ is any integer.}$$

Recall that an integer is any number from the list

$$0, \pm 1, \pm 2, \pm 3, \dots$$

CHAPTER 2. ANGLES AND RADIAN MEASURE

Question 0.6. The four rays shown in the diagram below are equally spaced. Describe all possible oriented angles with the indicated initial and terminal rays. What are the radian measures of these angles?



Solution:

One way of moving the initial ray to the terminal ray is to rotate counter-clockwise by one-quarter of a full revolution:

$$\frac{1}{4}(2\pi) = \frac{\pi}{2} \text{ radians.}$$

The other rotations that also do this are obtained by adding a number of full revolutions:

$$\frac{\pi}{4} + 2\pi n$$

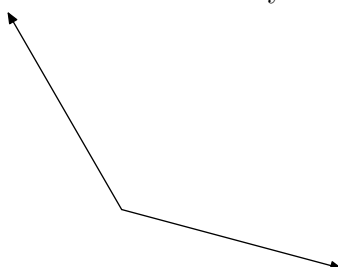
where n is an integer. Recall that an integer is any number from the list:

$$0, \pm 1, \pm 2, \dots$$

CHAPTER 3

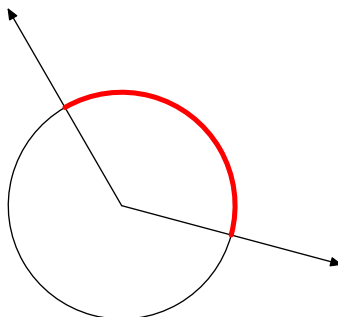
Unoriented Angles

An *unoriented* angle consists of two rays with a common vertex.



Such a configuration is simpler than an oriented angle since there is no implied rotation from one of the rays to the other.

There is a scheme for measuring unoriented angles. As before, draw a circle of *unit* radius centered at the vertex.



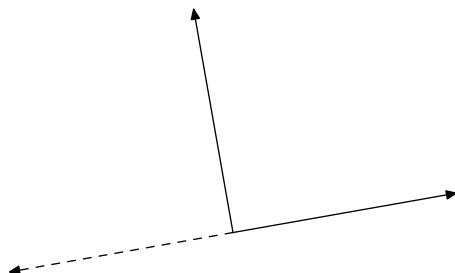
Note that the ray splits the circumference of the circle into two pieces. The radian measure of the unoriented ray is the length of the shorter of these two pieces. This procedure will always produce a number between 0 and π radians.

There are various common-sense geometric observations, having to do with unoriented angles, that are very important in trigonometry. These are treated in the rest of this section.

An unoriented angle will have value 0 radians if the two rays coincide.

If the two rays point in opposite directions, the measure will be π radians.

An unoriented angle of special importance is the *right angle*. Such an angle is constructed by bisecting an angle of π radians:

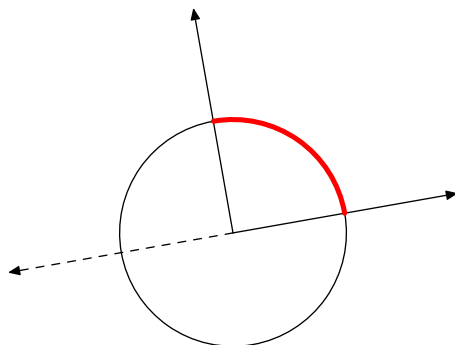


Question 0.7. What is the radian measure of the unoriented angle shown above?

CHAPTER 3. UNORIENTED ANGLES

Solution:

Draw a unit circle centered at the vertex. The shorter of the two arcs on this circle determines the radian measure of this unoriented angle.

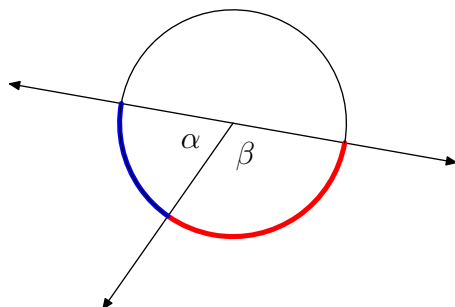


In this case, the length of the shorter arc is one-fourth of the total circumference:

$$\frac{1}{4}2\pi = \frac{\pi}{2}.$$

Therefore, any right angle has radian measure $\pi/2$.

Next compare the angles a given ray makes with another ray together with its opposite:



Note that the red and the blue arcs together give one-half of the circumference. Hence

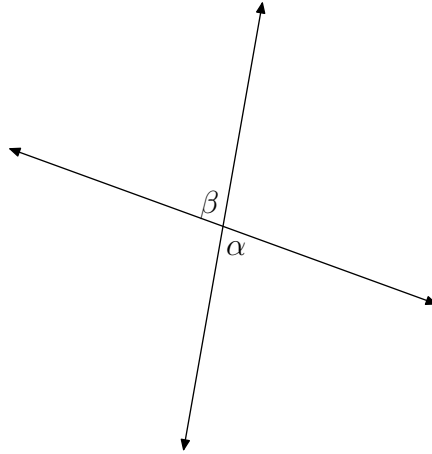
$$\alpha + \beta = \pi.$$

The following terminology is important. An unoriented angle is *acute* if it is less than $\pi/2$ radians. An unoriented angle is *obtuse* if it is greater than $\pi/2$ radians. Two unoriented angles whose radian measures add to π are called *supplementary*. Two unoriented angles whose radian measures add to $\pi/2$ are called *complementary*. Two unoriented angles whose radian measures are the same are called *congruent*.

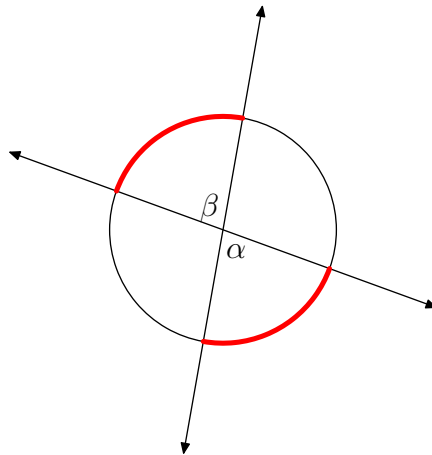
Hence, the two angles in the preceding figure are supplementary.

CHAPTER 3. UNORIENTED ANGLES

The two marked angles in the figure below are called *vertical angles*.

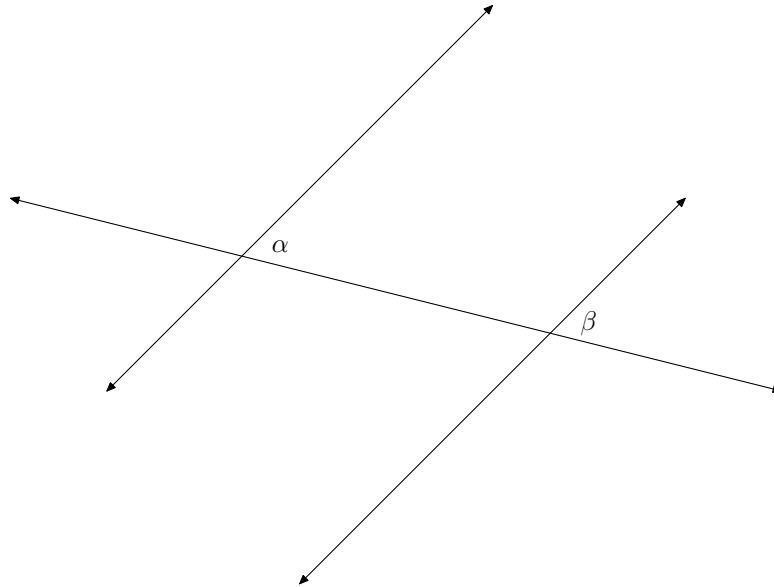


These two angles will determine two arcs on the circle of radius one centered at the point where the two lines intersect. It is visually obvious that these two arcs have the same length.

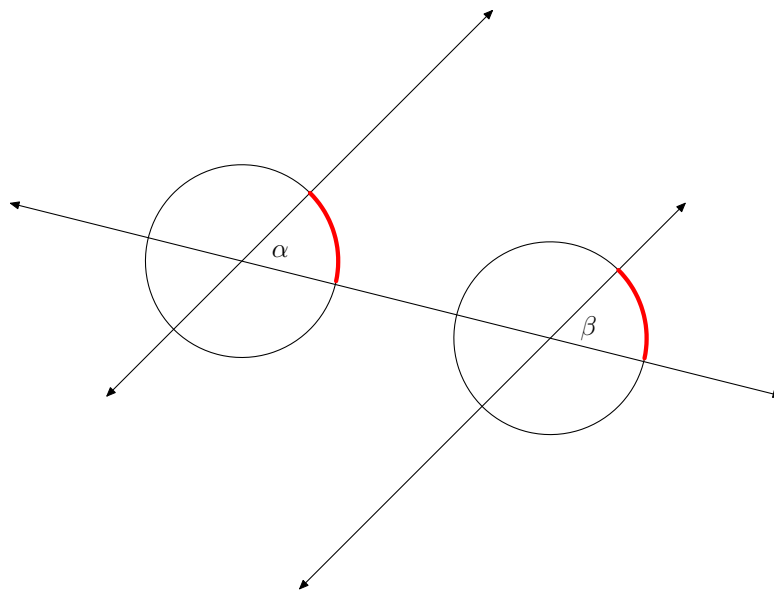


So, $\alpha = \beta$. The general principle here: vertical angles are congruent.

Another configuration that is important is where two parallel lines are both intersected by a third line called a transversal.



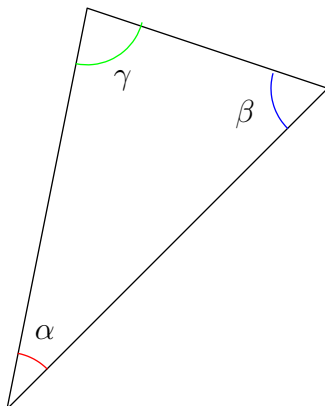
The two marked angles, are called *alternating angles*. To determine their radian measure, first draw circles of unit radii about their respective vertexes:



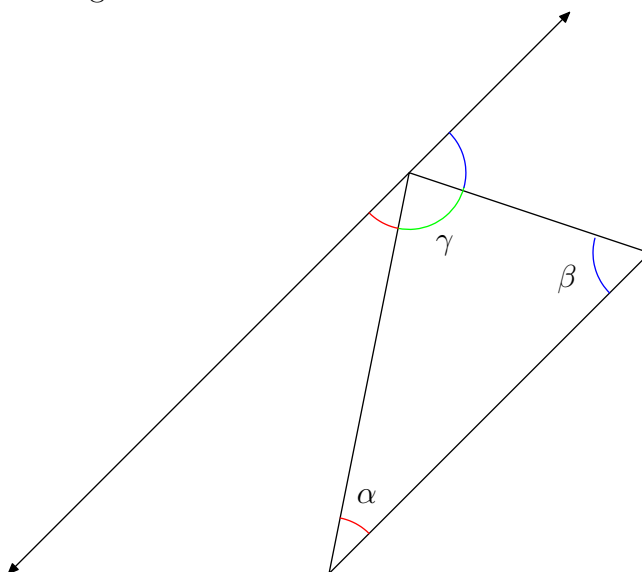
Each angle cuts off an arc. It is visually obvious that the lengths of these arcs are equal. As a consequence, these angles are congruent: $\alpha = \beta$. The general principle: alternating angles are congruent.

CHAPTER 3. UNORIENTED ANGLES

Consider now an arbitrary triangle:



Choose a vertex and draw a line through the vertex that is parallel to the opposite leg:

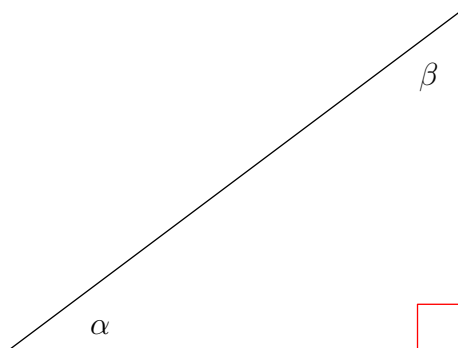


Our analysis of alternating angles tells us that the angles marked with the same color in this diagram are congruent. As a consequence, $\alpha + \beta + \gamma = \pi$. The general principle:

The three angles in a triangle sum to π radians.

!

Consider now a *right triangle*, i.e. a triangle one of whose angles had radian measure $\pi/2$.



Since the angles add up to π :

$$\alpha + \beta + \frac{\pi}{2} = \pi.$$

Subtracting $\pi/2$ from both sides,

$$\alpha + \beta = \frac{\pi}{2}.$$

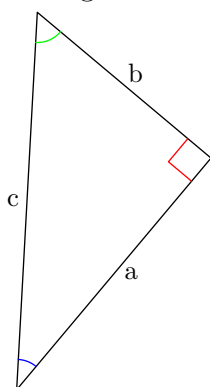
The general principle:

The acute angles in a right triangle are complementary.

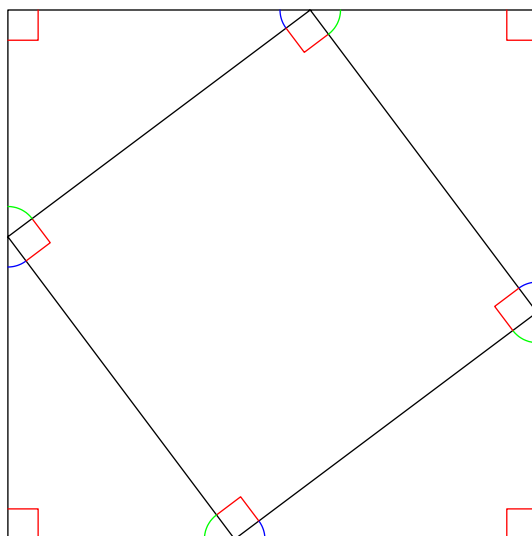
CHAPTER 4

The Pythagorean Theorem

Consider a typical right triangle as shown below.



The leg opposite the right angle is called the *hypotenuse*. Its length is labeled c . The lengths of the other two legs are labeled a and b . Imagine now four copies of this right triangle arranged as shown below.



The region as a whole is a square with sides of length $a+b$. Observe that this big square is comprised of four right triangles and an interior region. Since the acute angles in a right triangle are complementary, the four angles formed by the hypotenuses are all right angles. Hence

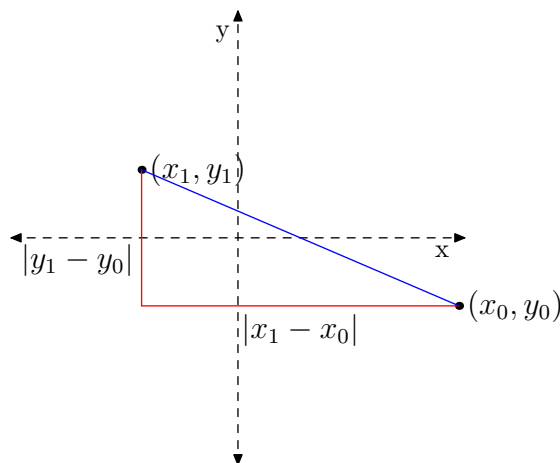
this interior regions is a square. Now, the area of the region is $(a + b)^2$ on the one hand and is the sum of the area of the five individual pieces on the other. The area of the interior square is $c \times c$ while the area of each of the right triangles is $ab/2$. Hence

$$\begin{aligned}(a + b)^2 &= 4 \frac{1}{2} ab + c^2 \\ a^2 + 2ab + b^2 &= 2ab + c^2\end{aligned}$$

Subtracting $2ab$ from both sides yields the *Pythagorean theorem*:

$$a^2 + b^2 = c^2$$

The first important consequence of the Pythagorean formula is the *distance formula*. Consider two points P_0 and P_1 in the co-ordinate plane with co-ordinates (x_0, y_0) and (x_1, y_1) , respectively.



The line segment that determines the distance between the points is the hypotenuse of a right triangle with legs of length $|x_1 - x_0|$ and $|y_1 - y_0|$. If D denotes the distance between P_0 and P_1 ,

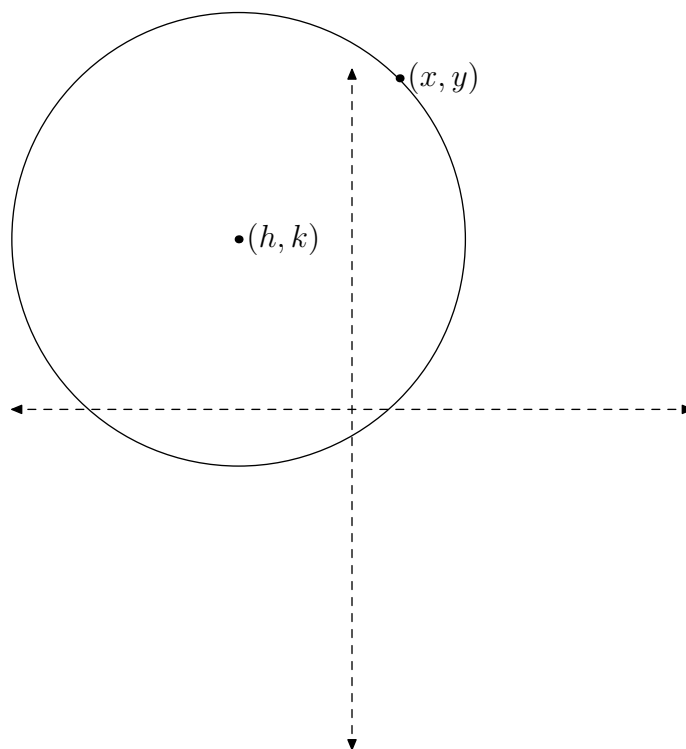
$$\begin{aligned}D^2 &= |x_1 - x_0|^2 + |y_1 - y_0|^2 \\ D &= \sqrt{|x_1 - x_0|^2 + |y_1 - y_0|^2}\end{aligned}$$

Since $|x|^2$ is the same as x^2 ,

$$D = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$

This is the *distance formula*.

The distance formula can be used to determine the algebraic equations of circles. Consider the circle with radius r and center with co-ordinates (h, k) .



If the point with co-ordinates (x, y) is on this circle, its distance to the center is r . Because of the distance formula,

$$r^2 = (x - h)^2 + (y - k)^2.$$

This is the equation for the circle of radius r and center (h, k) .

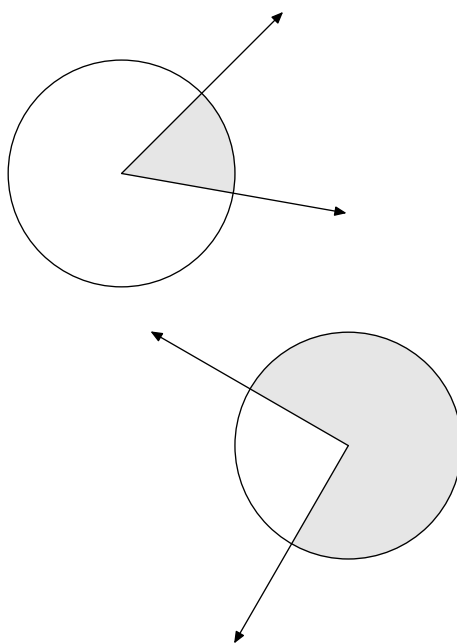
An especially important circle in trigonometry is the circle of radius one and center at the origin $(0, 0)$. The equation for this *unit circle* is

$$x^2 + y^2 = 1.$$

CHAPTER 5

Sectors

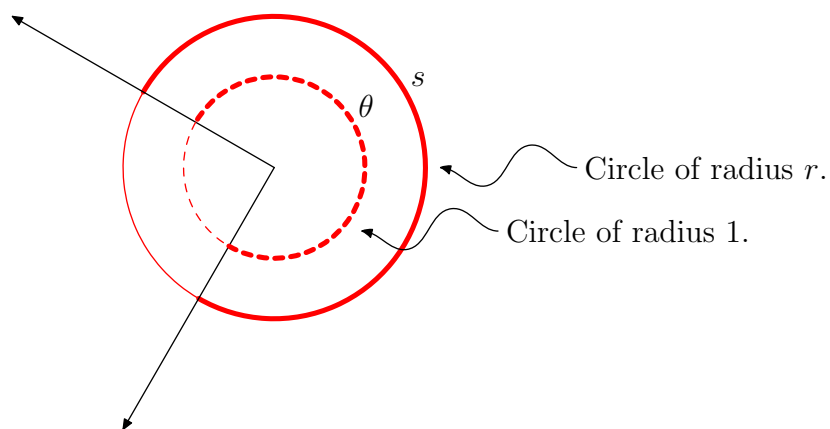
A *sector* is a region enclosed by a circle, and two rays with vertexes at the center of the circle. Examples of such regions are show below.



Note this circle is not assumed to have radius one.

A sector is described by two quantities: the radius of the circle and the radian measure of the angle that determines the gap between the two rays. Write r for the radius of the circle and θ for the radian measure of the angle determined by the two rays. Given this information, how are the arc-length and area of the sector determined?

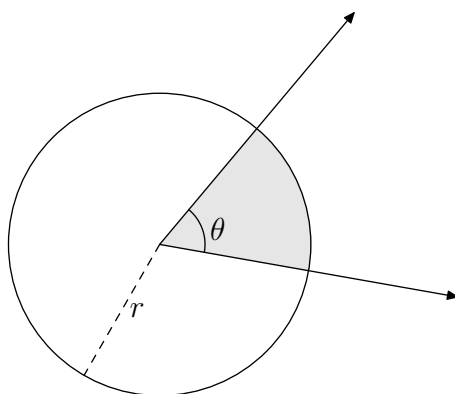
First note that the radian measure is just the arc length along the *circle of radius one* centered at the common vertex of the rays. If $r = 2$, the arc length will double, if $r = 3$, the arc length will triple, and so on.



Hence the arc length s is given by the formula

$$s = r\theta.$$

The *area* enclosed by a sector is calculated by a different strategy: one determines the fraction of the entire circle determined by the sector at hand.



Since the full circle corresponds to 2π radians, the sector is the following fraction of the entire circle:

$$\frac{\theta}{2\pi}.$$

The area of the full circle of radius r is πr^2 . As a consequence, the area of the sector is

$$\frac{\theta}{2\pi} \pi r^2 = \frac{1}{2} \theta r^2.$$

Therefore the area of a sector of θ radians is

$$A = \frac{1}{2} \theta r^2.$$

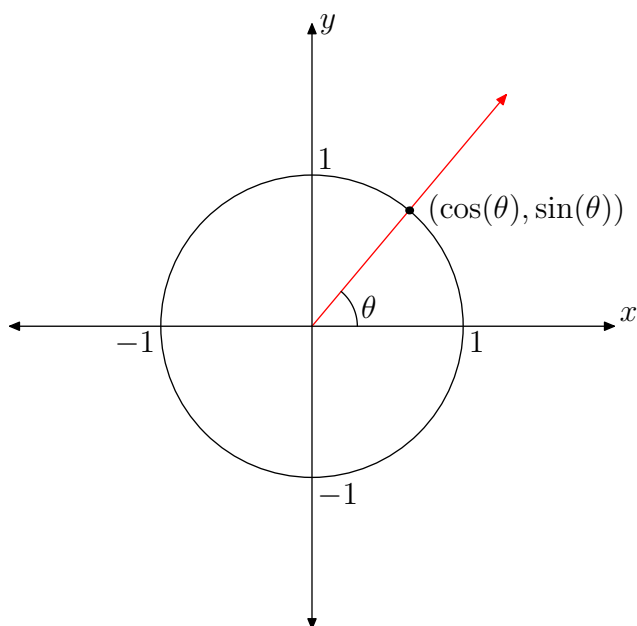
Part 2

The Six Circular Functions

CHAPTER 6

The Unit Circle Definitions

The *unit circle*, as mentioned previously, is the circle of radius one and center $(0,0)$. An angle is in *standard position* if its initial ray coincides with the positive x -axis. For any real number θ , draw an angle in standard position with θ as its radian measure.



The terminal ray for this angle intersects the unit circle at a well determined point. The co-ordinates for this point will be labeled as

$$(\cos(\theta), \sin(\theta)).$$

This procedure defines two values for any real number θ . These are called the *cosine* and *sine* functions: $\cos(\theta)$ and $\sin(\theta)$.

There are four other associated functions:

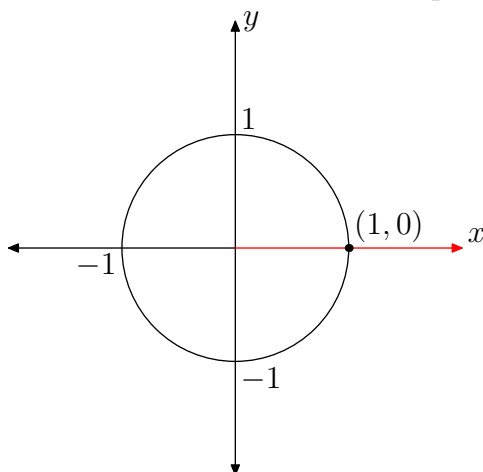
$$\begin{aligned} \sec(\theta) &= \frac{1}{\cos(\theta)} & \csc(\theta) &= \frac{1}{\sin(\theta)} \\ \tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} & \cot(\theta) &= \frac{\cos(\theta)}{\sin(\theta)} \end{aligned}$$

CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Question 0.8. Calculate the values of the six trigonometric functions when $\theta = 0$ radians.

Solution:

Begin by drawing the standard angle with $\theta = 0$. In this case, both the initial and terminal ray coincide with the positive x -axis. Therefore the terminal ray intersects the unit circle at the point $(1, 0)$.



Hence,

$$\cos(0) = 1 \quad \text{and} \quad \sin(0) = 0.$$

Based on this, the values of the other four trigonometric functions at $\theta = 0$ are:

$\sec(0) = 1$	$\csc(0) : \text{undefined}$
$\tan(0) = 0$	$\cot(0) : \text{undefined}$

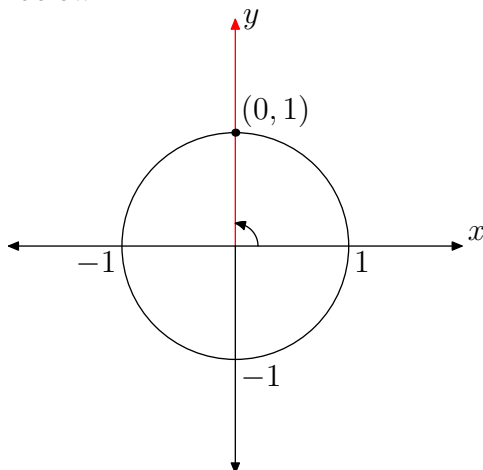
CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Question 0.9. Calculate the values of the six trigonometric functions when $\theta = \pi/2$ radians.

CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Solution:

Since $\pi/2 = 1/4 \cdot 2\pi$, this angle is one-quarter of a full revolution in the counter-clockwise orientation. The standard angle with this radian measure is shown below.



The terminal ray intersects the unit circle at the point $(0, 1)$. Hence,

$$\cos(\pi/2) = 0 \quad \text{and} \quad \sin(\pi/2) = 1.$$

Based on this, the values of the other four trigonometric functions at $\theta = \pi/2$ are:

$\sec(\pi/2) : \text{undefined}$	$\csc(\pi/2) = 1$
$\tan(\pi/2) : \text{undefined}$	$\cot(\pi/2) = 0$

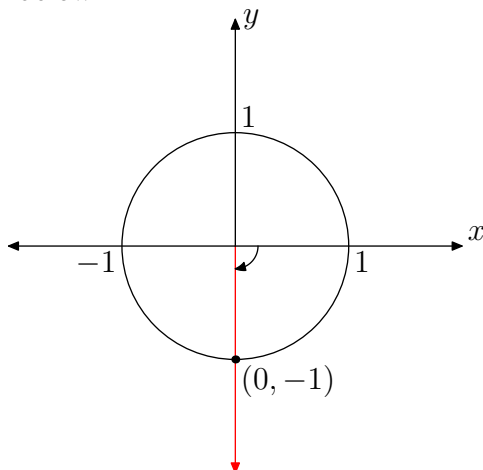
CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Question 0.10. Calculate the values of the six trigonometric functions when $\theta = -\pi/2$ radians.

CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Solution:

Since $-\pi/2 = -1/4 \cdot 2\pi$, this angle is one-quarter of a full revolution in the clockwise orientation. The standard angle with this radian measure is shown below.



The terminal ray intersects the unit circle at the point $(0, -1)$. Hence,

$$\cos(-\pi/2) = 0 \quad \text{and} \quad \sin(-\pi/2) = -1.$$

Based on this, the values of the other four trigonometric functions at $\theta = -\pi/2$ are:

$\sec(-\pi/2) : \text{undefined}$	$\csc(-\pi/2) = -1$
$\tan(-\pi/2) : \text{undefined}$	$\cot(-\pi/2) = 0$

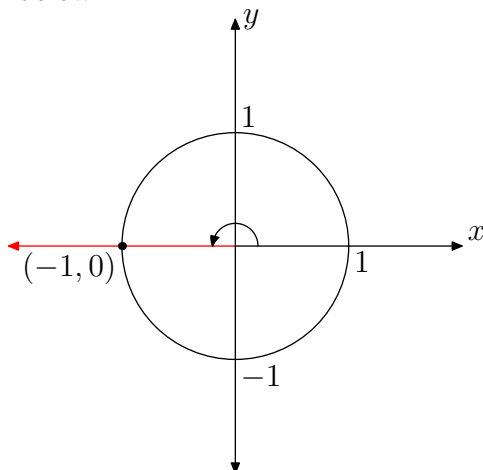
CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Question 0.11. Calculate the values of the six trigonometric functions when $\theta = \pi$ radians.

CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Solution:

Since $\pi = 1/2 \cdot 2\pi$, this angle is one-half of a full revolution in the counter-clockwise orientation. The standard angle with this radian measure is shown below.



The terminal ray intersects the unit circle at the point $(-1, 0)$. Hence,

$$\cos(\pi) = -1 \quad \text{and} \quad \sin(\pi) = 0.$$

Based on this, the values of the other four trigonometric functions at $\theta = \pi$ are:

$\sec(\pi) = -1$	$\csc(\pi) : \text{undefined}$
$\tan(\pi) = 0$	$\cot(\pi) : \text{undefined}$

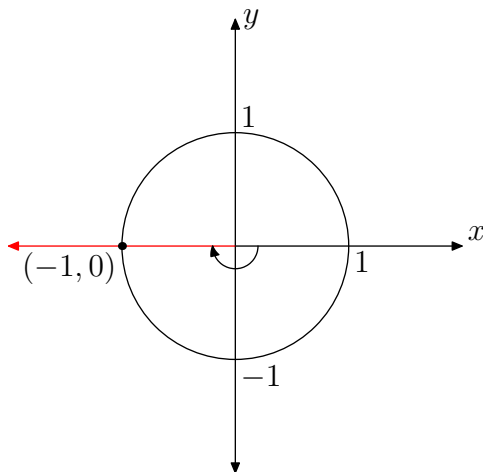
CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Question 0.12. Calculate the values of the six trigonometric functions when $\theta = -\pi$ radians.

CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Solution:

Since $-\pi = -1/2 \cdot 2\pi$, this angle is one-half of a full revolution in the clockwise orientation. The standard angle with this radian measure is shown below.



The terminal ray intersects the unit circle at the point $(-1,0)$. Hence,

$$\cos(-\pi) = -1 \quad \text{and} \quad \sin(-\pi) = 0.$$

Based on this, the values of the other four trigonometric functions at $\theta = -\pi$ are:

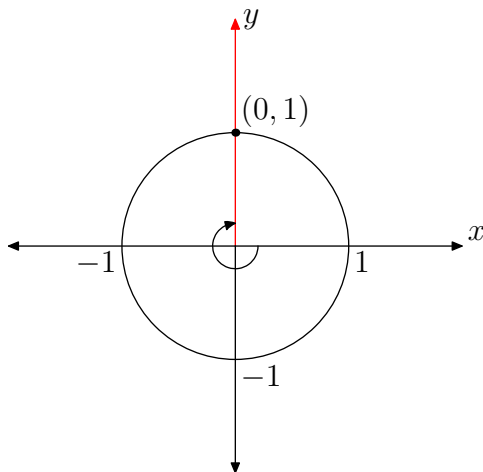
$$\begin{array}{ll} \sec(-\pi) = -1 & \csc(-\pi) : \text{undefined} \\ \tan(-\pi) = 0 & \cot(-\pi) : \text{undefined} \end{array}$$

CHAPTER 6. THE UNIT CIRCLE DEFINITIONS

Question 0.13. Calculate the values of the six trigonometric functions when $\theta = -3\pi/2$ radians.

Solution:

This angle, $\theta = -3/4 \cdot 2\pi$ is three-fourths of a full revolution with the clockwise orientation. The standard angle with this radian measure is shown below.



The terminal ray intersects the unit circle at the point $(0, 1)$. Hence,

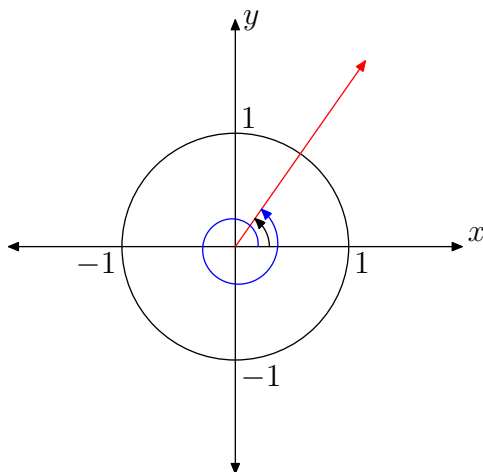
$$\cos(-3\pi/2) = 0 \quad \text{and} \quad \sin(-3\pi/2) = 1.$$

Based on this, the values of the other four trigonometric functions at $\theta = -\pi$ are:

$$\begin{array}{ll} \sec(-3\pi/2) : \text{undefined} & \csc(-3\pi/2) = 1 \\ \tan(-3\pi/2) : \text{undefined} & \cot(-3\pi/2) = 0 \end{array}$$

This section concludes with a few general observations about the trigonometric functions.

To begin with, note that the terminal ray for an angle θ and $\theta + 2\pi$ are the same. Hence, the point where they intersect the unit circle is the same.



Hence the order pairs $(\cos(\theta), \sin(\theta))$ and $(\cos(\theta + 2\pi), \sin(\theta + 2\pi))$ are the same. Therefore,

$$\begin{aligned}\cos(\theta + 2\pi) &= \cos(\theta) \\ \sin(\theta + 2\pi) &= \sin(\theta)\end{aligned}$$

for all angles θ .

Note that this type of reasoning still works if the 2π subtracted instead: *i.e.* the terminal rays for θ and $\theta - 2\pi$ are the same. Hence, the order pairs $(\cos(\theta), \sin(\theta))$ and $(\cos(\theta - 2\pi), \sin(\theta - 2\pi))$ are the same. Therefore,

$$\begin{aligned}\cos(\theta + 2\pi) &= \cos(\theta) \\ \sin(\theta + 2\pi) &= \sin(\theta)\end{aligned}$$

for all angles θ .

Indeed, the result still holds if any number of full revolutions are added or subtracted from θ , *i.e.*

$$\begin{aligned}\cos(\theta + 2\pi n) &= \cos(\theta) \\ \sin(\theta + 2\pi n) &= \sin(\theta)\end{aligned}$$

for all θ , where n is any integer. Since the other four trigonometric functions are built up out of sine and cosine, they too will satisfy similar identities:

$$\sec(\theta + 2\pi n) = \sec(\theta)$$

$$\csc(\theta + 2\pi n) = \csc(\theta)$$

$$\tan(\theta + 2\pi n) = \tan(\theta)$$

$$\cot(\theta + 2\pi n) = \cot(\theta)$$

for all θ , where n is any integer.

Another important set of identities, arises from the observation that the ordered pair $(\cos(\theta), \sin(\theta))$ is on the unit circle with equation $x^2 + y^2 = 1$. Therefore,

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

for all θ . Dividing both sides of the above identity by $\cos^2(\theta)$ yields:

$$1 + \frac{\sin^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$1 + \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^2 = \left(\frac{1}{\cos(\theta)}\right)^2$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

Division by $\sin^2(\theta)$ instead, produces

$$\frac{\cos^2(\theta)}{\sin^2(\theta)} + 1 = \frac{1}{\sin^2(\theta)}$$

$$\left(\frac{\cos(\theta)}{\sin(\theta)}\right)^2 + 1 = \left(\frac{1}{\sin(\theta)}\right)^2$$

$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

These three identities are called the *Pythagorean identities*:

$$\cos^2(\theta) + \sin^2(\theta) = 1$$

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

$$\cot^2(\theta) + 1 = \csc^2(\theta)$$

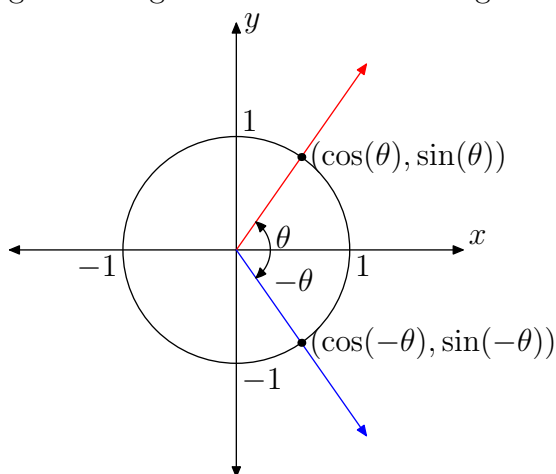
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CHAPTER 7

Symmetry

The symmetry inherent in the unit circle gives rise to several crucial identities. This section describes this kind of reasoning.

Consider a general angle θ and the related angle $-\theta$.



The terminal rays will be symmetric about the horizontal axis. Hence the points with co-ordinates $(\cos(\theta), \sin(\theta))$ and $(\cos(-\theta), \sin(-\theta))$ will be reflections of each other about the x -axis. So the x -coordinates of these two points must be the same while the y -coordinates must be opposites of each other:

$$\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta)$$

for all θ .

Question 0.14. Derive the corresponding identities for sec, csc, tan and cot.

Solution:

Using the definitions of these trigonometric functions,

$$\begin{aligned}\sec(-\theta) &= \frac{1}{\cos(-\theta)} \\ &= \frac{1}{\cos(\theta)} = \sec(\theta).\end{aligned}$$

The others are derived similarly. For example,

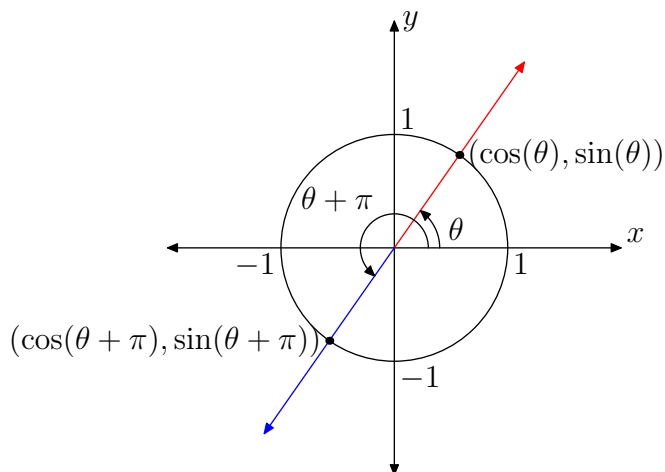
$$\begin{aligned}\tan(-\theta) &= \frac{\sin(-\theta)}{\cos(-\theta)} \\ &= \frac{-\sin(\theta)}{\cos(\theta)} = -\tan(\theta).\end{aligned}$$

The two remaining identities are done similarly.

In summary,

$$\begin{aligned}\sec(-\theta) &= \sec(\theta) \\ \csc(-\theta) &= -\csc(\theta) \\ \tan(-\theta) &= -\tan(\theta) \\ \cot(-\theta) &= -\cot(\theta).\end{aligned}$$

Next, compare the terminal rays for a general angle θ and the related angle $\theta + \pi$.



The terminal rays here are symmetric through the origin. Thus the co-ordinates of the points $(\cos(\theta), \sin(\theta))$ and $(\cos(\theta + \pi), \sin(\theta + \pi))$ are opposites: !

$$\cos(\theta + \pi) = -\cos(\theta)$$

$$\sin(\theta + \pi) = -\sin(\theta)$$

for all angles θ .

Question 0.15. What are the corresponding identities for the other four trigonometric functions?

Solution:

Here is a sample calculation:

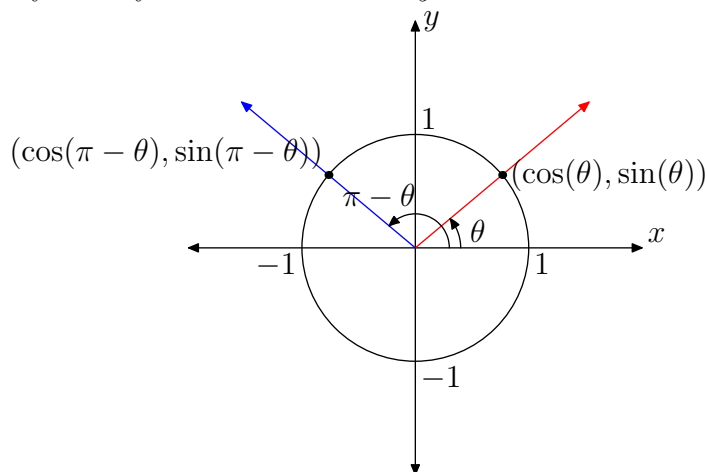
$$\begin{aligned}\tan(\theta + \pi) &= \frac{\sin(\theta + \pi)}{\cos(\theta + \pi)} \\ &= \frac{-\sin(\theta)}{-\cos(\theta)} \\ &= \frac{\sin(\theta)}{\cos(\theta)} \\ &= \tan(\theta).\end{aligned}$$

The three remaining cases are left to the reader.

In summary,

$$\begin{aligned}\sec(\theta + \pi) &= -\sec(\theta) \\ \csc(\theta + \pi) &= -\csc(\theta) \\ \tan(\theta + \pi) &= \tan(\theta) \\ \cot(\theta + \pi) &= \cot(\theta).\end{aligned}$$

Next: how is a typical angle θ related to $\pi - \theta$? In this case, the terminal rays are symmetric about the y -axis.



As a consequence, the points $(\cos(\theta), \sin(\theta))$ and $(\cos(\pi - \theta), \sin(\pi - \theta))$ are related by the same symmetry. Hence, !

$$\cos(\pi - \theta) = -\cos(\theta)$$

$$\sin(\pi - \theta) = \sin(\theta)$$

for all angles θ .

Question 0.16. Derive the corresponding identities for the other four trigonometric functions.

Solution:

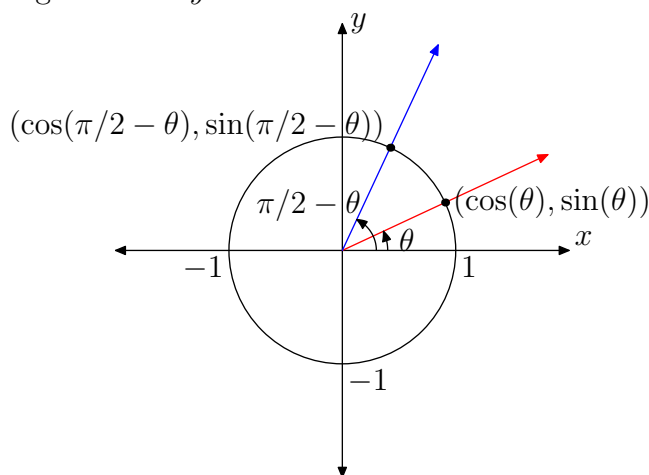
Here is the calculation for \cot :

$$\begin{aligned}\cot(\pi - \theta) &= \frac{\cos(\pi - \theta)}{\sin(\pi - \theta)} \\ &= \frac{-\cos(\theta)}{\sin(\theta)} \\ &= -\cot(\theta)\end{aligned}$$

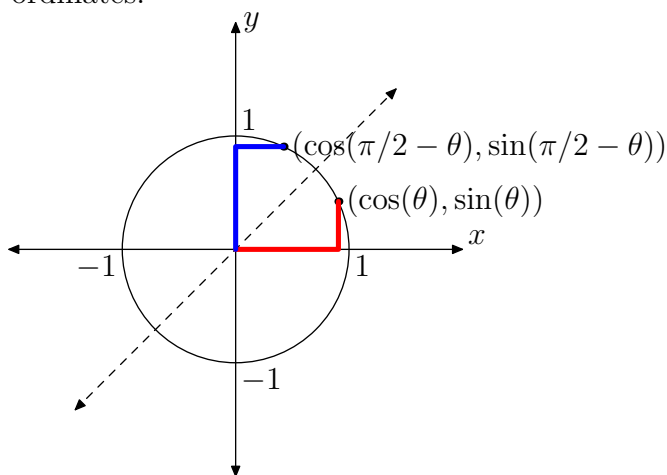
The other three calculations work much the same way: In summary:

$$\begin{aligned}\sec(\pi - \theta) &= -\sec(\theta) \\ \csc(\pi - \theta) &= \csc(\theta) \\ \tan(\pi - \theta) &= -\tan(\theta) \\ \cot(\pi - \theta) &= -\cot(\theta).\end{aligned}$$

Next, compare a typical angle θ with the angle $\pi/2 - \theta$. The terminal ray for θ is obtained by rotating the positive x -axis by θ radians, while the terminal ray for $\pi/2 - \theta$ is obtained by rotating the positive y -axis the same amount in the opposite orientation. The symmetry here is about the diagonal line: $y = x$.



The points $(\cos(\theta), \sin(\theta))$ and $(\cos(\pi/2 - \theta), \sin(\pi/2 - \theta))$ are symmetric about this line. Now reflection about the $y = x$ line interchanges the two co-ordinates.



Therefore the following pair of identities hold for all angles θ :

$$\cos(\pi/2 - \theta) = \sin(\theta)$$

$$\sin(\pi/2 - \theta) = \cos(\theta)$$

Question 0.17. Derive the corresponding identities for the other four trigonometric functions.

Solution:

Here are the calculations for \tan and \sec :

$$\begin{aligned}\tan(\pi/2 - \theta) &= \frac{\sin(\pi/2 - \theta)}{\cos(\pi/2 - \theta)} \\ &= \frac{\cos(\theta)}{\sin(\theta)} \\ &= \cot(\theta).\end{aligned}$$

$$\begin{aligned}\sec(\pi/2 - \theta) &= \frac{1}{\cos(\pi/2 - \theta)} \\ &= \frac{1}{\sin(\theta)} \\ &= \csc(\theta).\end{aligned}$$

The other two identities are derived in a similar way. In summary,

$$\begin{aligned}\sec(\pi/2 - \theta) &= \csc(\theta) \\ \csc(\pi/2 - \theta) &= \sec(\theta) \\ \tan(\pi/2 - \theta) &= \cot(\theta) \\ \cot(\pi/2 - \theta) &= \tan(\theta).\end{aligned}$$

CHAPTER 8

Graphs

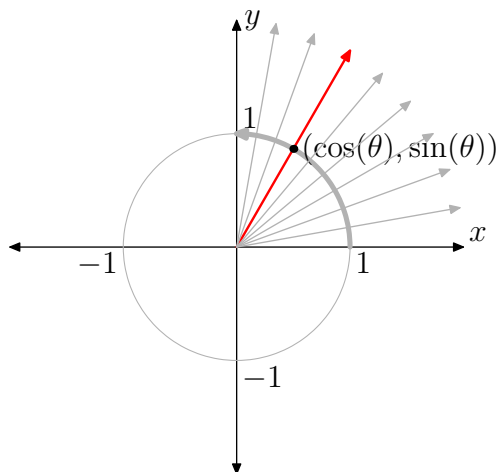
A function is a rule that assigns a value to any number in the domain of the function. A function is usually defined algebraically by means of a formula. However, it can also be defined geometrically. This is the case with the six trigonometric functions that were defined in terms of the unit circle scheme discussed in the previous sections.

The defining rule for a function can also be encoded in terms of its graph. This is a plot of all ordered pairs of the form

(input number, output number)

as the input number goes through the domain of the function. In this section, the graphs of the trigonometric functions are described and related to the unit circle picture of the previous sections.

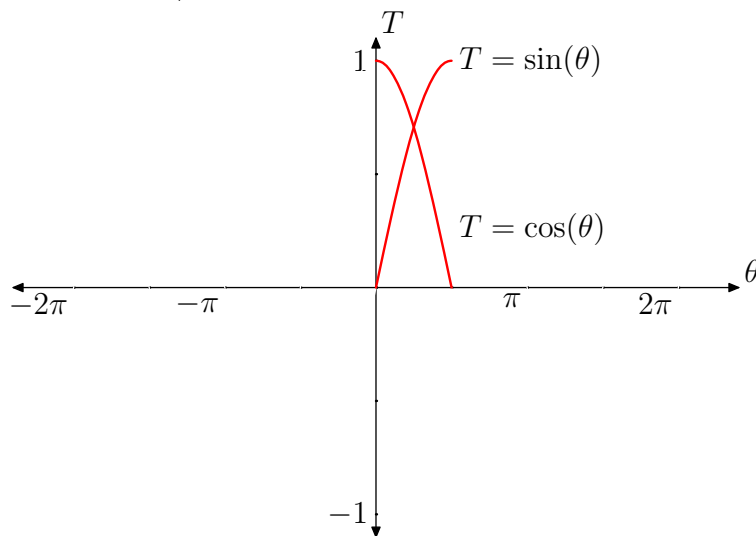
Imagine the angle θ moving slowly from 0 to $\pi/2$ radians. The terminal ray for θ would then rotate from the positive x -axis to the positive y -axis as shown below:



As this happens, the ordered pair $(\cos(\theta), \sin(\theta))$ moves along the unit circle from $(1, 0)$ to $(0, 1)$. Hence, as θ goes from 0 to $\pi/2$:

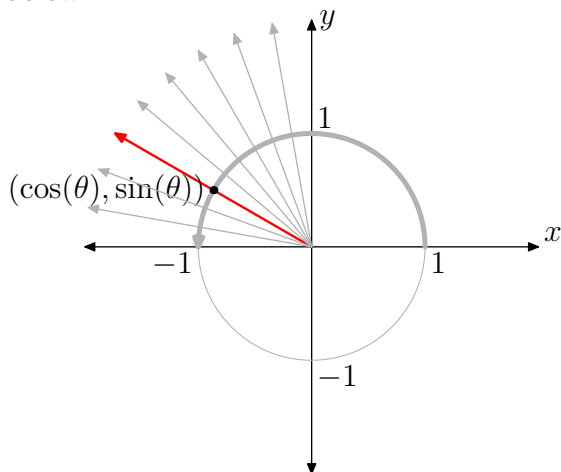
- $\cos(\theta)$ goes from 1 down to 0, and
- $\sin(\theta)$ goes from 0 up to 1.

This determines the behavior of the graphs $T = \cos(\theta)$ and $T = \sin(\theta)$ from $\theta = 0$ to $\theta = \pi/2$, as highlighted below:



CHAPTER 8. GRAPHS

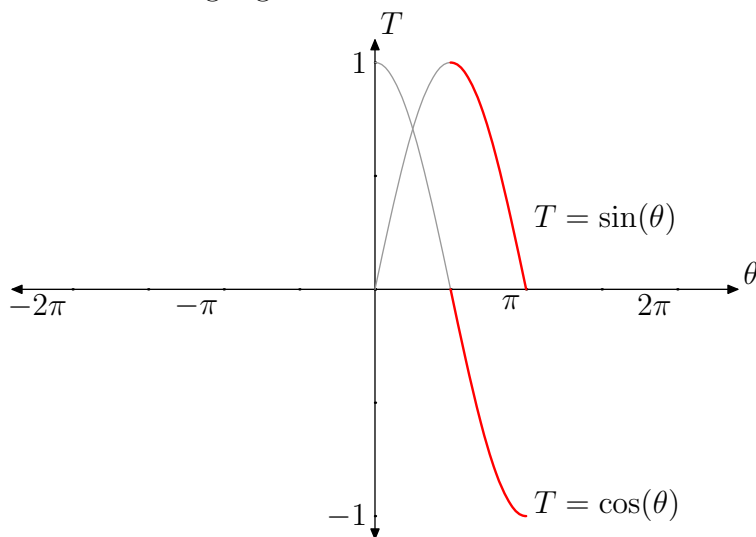
Next, imagine θ moving slowly from $\pi/2$ to π radians. The terminal ray for θ would then rotate from the positive y -axis to the negative x -axis as shown below:



As this happens, the ordered pair $(\cos(\theta), \sin(\theta))$ moves along the unit circle from $(0, 1)$ to $(-1, 0)$. Hence, as θ goes from $\pi/2$ to π :

- $\cos(\theta)$ goes from 0 down to -1 , and
- $\sin(\theta)$ goes from 1 down to 0.

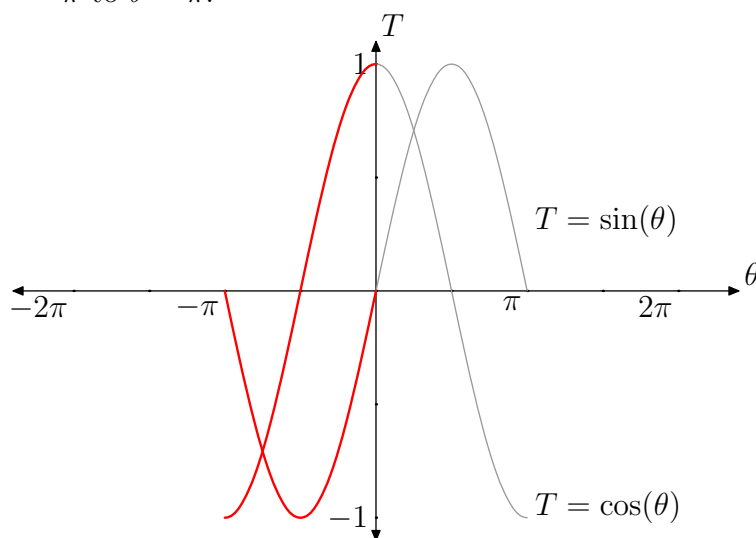
The implied behavior of the graphs $T = \cos(\theta)$ and $T = \sin(\theta)$ from $\theta = \pi/2$ to $\theta = \pi$ is highlighted below:



So, at this point, the graphs of \cos and \sin are evident on the interval from $\theta = 0$ to $\theta = \pi$. The rest can be obtained by using the identities that were proved in the last section. Recall, first of all, that

$$\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta)$$

for all angles θ . So the graph of \cos is symmetric about the vertical axis while the graph of \sin is symmetric through the origin. This determines the graphs of \cos and \sin from $\theta = 0$ to $\theta = -\pi$. We extend the graph of \cos by reflecting the known part about the vertical axis and the graph of \sin about the origin. This gives us the graphs on the interval from $\theta = -\pi$ to $\theta = \pi$:

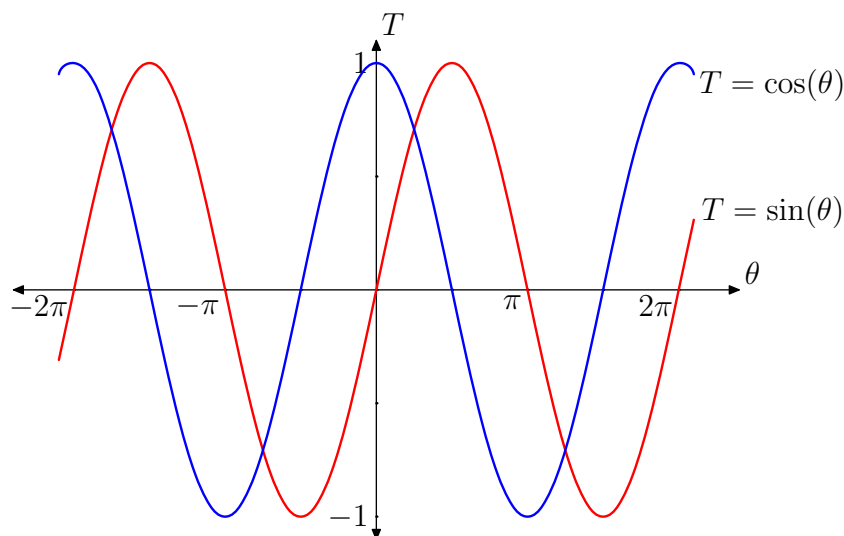


Now, recall the identities

$$\cos(\theta + 2\pi n) = \cos(\theta) \quad \text{and} \quad \sin(\theta + 2\pi n) = \sin(\theta)$$

that hold for all angles θ . These imply that the graphs of \cos and \sin repeat themselves every 2π units along the θ -axis. Thus, the remaining portions of these graphs can be obtained by shifting the known pieces to the left and right by multiples of 2π .

This leads to the graphs shown below:



The other four trigonometric functions are built up out of sine and cosine. The shape of their graphs can also be deduced because of this.

Consider $T = \tan(\theta)$. The definition of this function is a fraction:

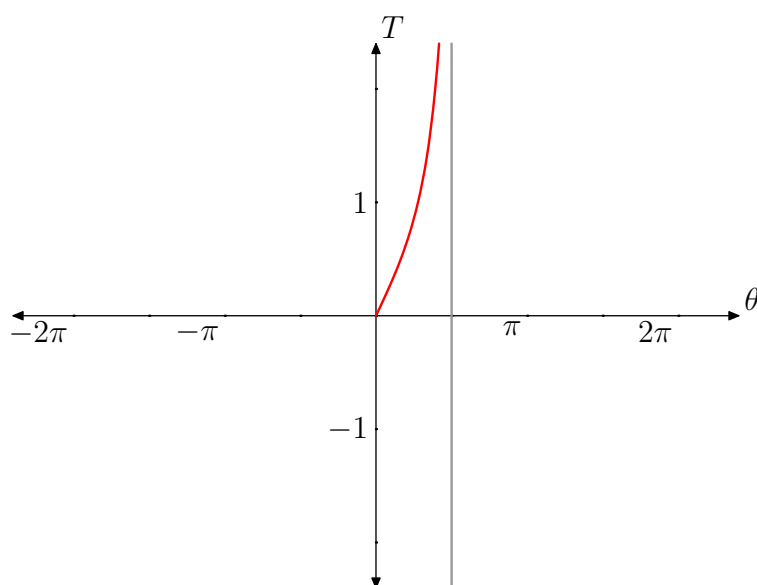
$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}.$$

What is the behavior of the entire fraction as θ moves slowly from 0 to $\pi/2$?

When θ is a small positive number, the numerator ($\sin(\theta)$) is a small positive number while the denominator ($\cos(\theta)$) is close to one. Hence, the fraction $\sin(\theta)/\cos(\theta)$ is very close to $0/1 = 0$.

As θ slowly increases to $\pi/2$, The numerator ($\sin(\theta)$) is increasing up to one while the denominator ($\cos(\theta)$) is decreasing to a value of zero. By the time θ is almost up to $\pi/2$, the numerator of $\frac{\sin(\theta)}{\cos(\theta)}$ is very close to one while the denominator is an extremely small positive number. As a consequence, the fraction itself must be a very large number. In fact, the closer θ is $\pi/2$ the larger the magnitude of $\tan(\theta)$.

The preceding analysis can be summarized as follows: *as θ goes from 0 to $\pi/2$, $\tan(\theta)$ goes from 0 to infinity.* It determines the shape of the graph of $T = \tan(\theta)$ for $\theta = 0$ to $\theta = \pi/2$.

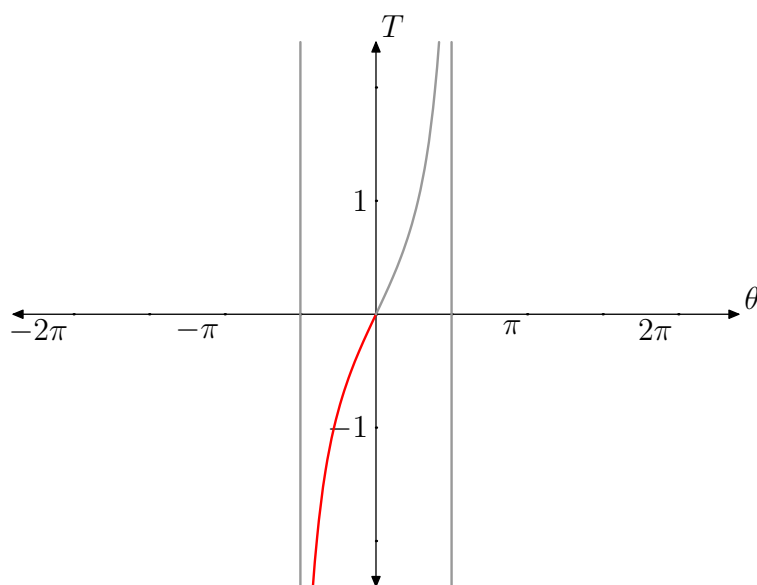


Note that the behavior as θ gets close to $\pi/2$ creates a vertical asymptote at $x = \pi/2$ (indicated by the gray vertical line).

The rest of the graph can be deduced from various known identities. First, recall that

$$\tan(-\theta) = -\tan(\theta)$$

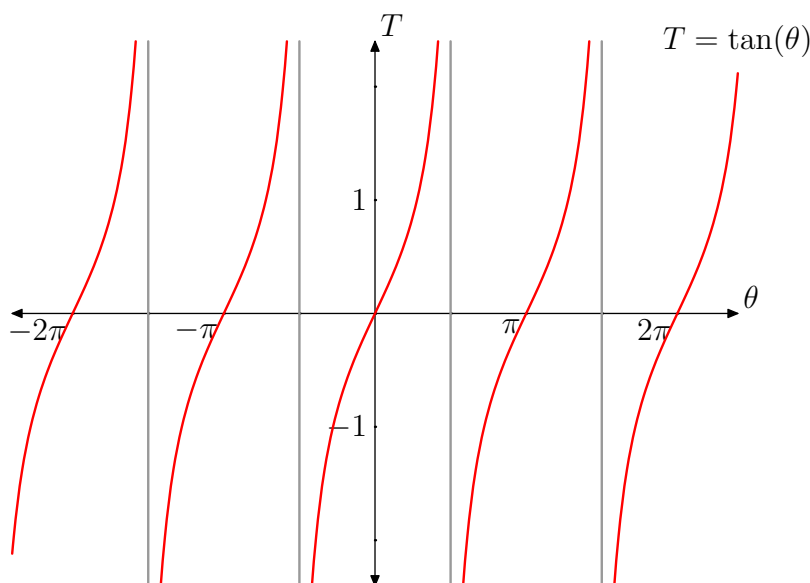
for all θ . Hence the graph of \tan is symmetric through the origin. Because of this, the graph of \tan from $\theta = -\pi/2$ to $\theta = 0$ can be determined:



A second identity for \tan can now be applied:

$$\tan(\theta + \pi) = \tan(\theta)$$

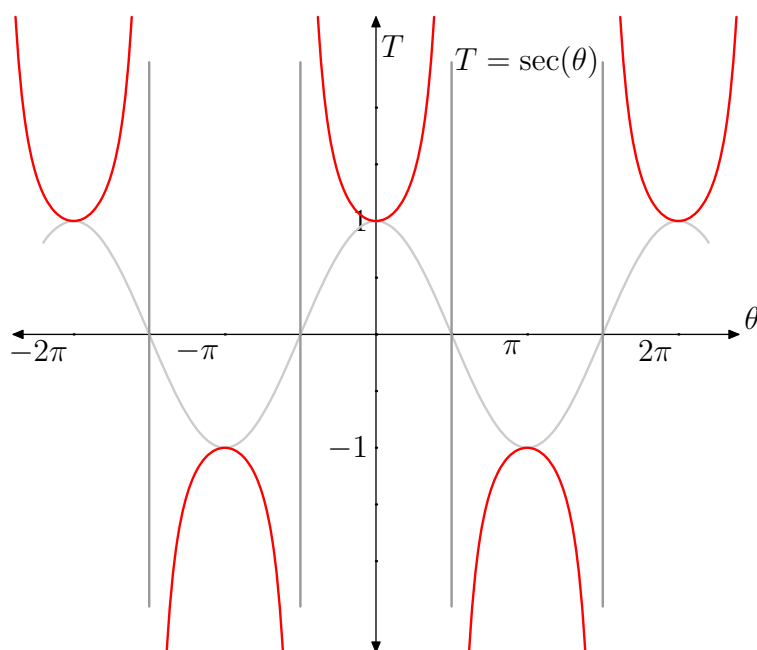
for all θ . Hence, \tan is periodic with period π . Because of this, the rest of the graph of \tan can be obtained by repeatedly shifting the piece drawn above to the left and right by integer multiples of π . This produces the following graph. !



Next, consider $T = \sec(\theta)$ which is defined as

$$\sec(\theta) = \frac{1}{\cos(\theta)}.$$

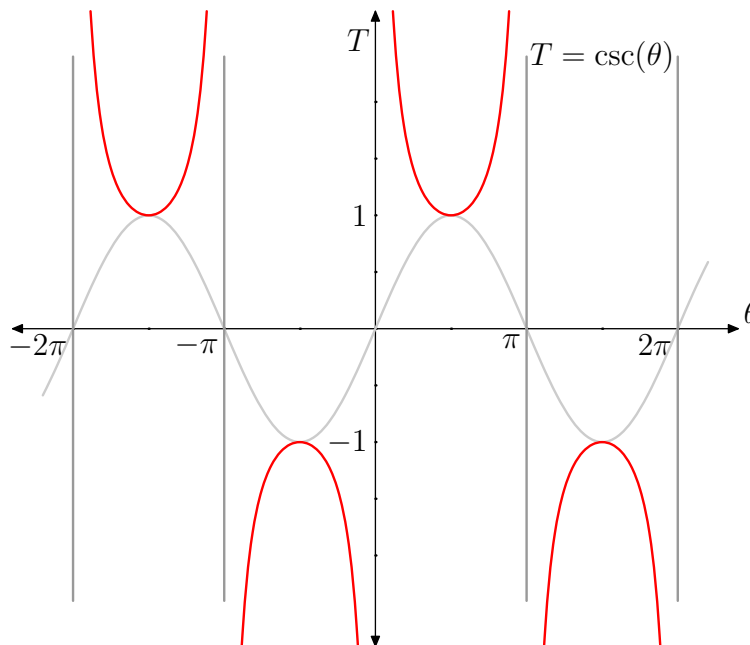
If the denominator in this fraction is small in size and positive, the reciprocal is large in magnitude and positive. On the other hand, if the denominator is small in size and negative, the reciprocal is large in magnitude and negative. Hence $\sec(\theta)$ will have vertical asymptotes at the roots of $\cos(\theta)$. At angles where $\cos(\theta) = \pm 1$ $\sec(\theta) = 1/(\pm 1) = \pm 1$. If $0 < \cos(\theta) < 1$, the reciprocal $\sec(\theta)$ will be positive and *larger* than one. If $0 > \cos(\theta) > -1$, the reciprocal $\sec(\theta)$ will be negative and *less* than minus one. By putting this all together, we obtain a graph of the following shape:



Question 0.18. Sketch the graph of the functions $\csc(\theta)$ and $\cot(\theta)$.

Solution:

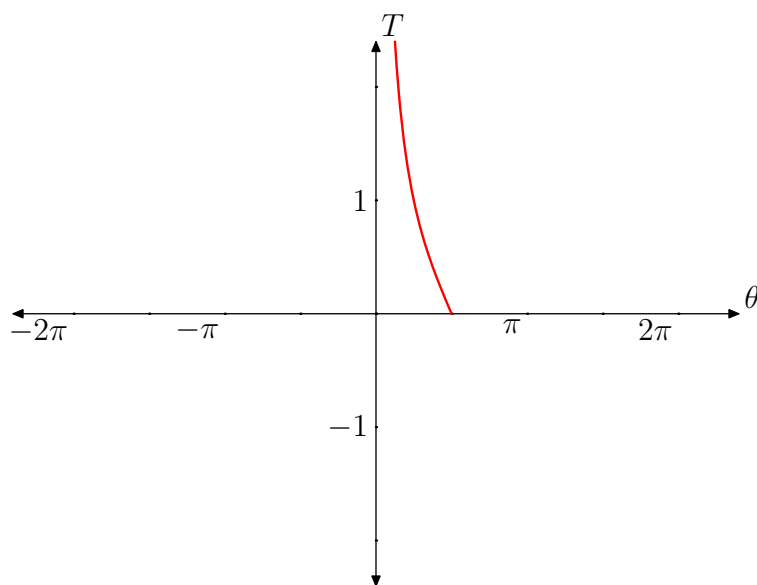
The analysis of $\csc(\theta) = 1/\sin(\theta)$ is similar to that of $\sec(\theta)$, except that it is based on the graph of $\sin(\theta)$. The final plot has the following form:



The analysis of $\cot(\theta) = \cos(\theta)/\sin(\theta)$ is similar to that of $\tan(\theta)$, except that the roles of \sin and \cos are reversed. So, when θ is a small positive angle the denominator of the fraction

$$\frac{\cos(\theta)}{\sin(\theta)}$$

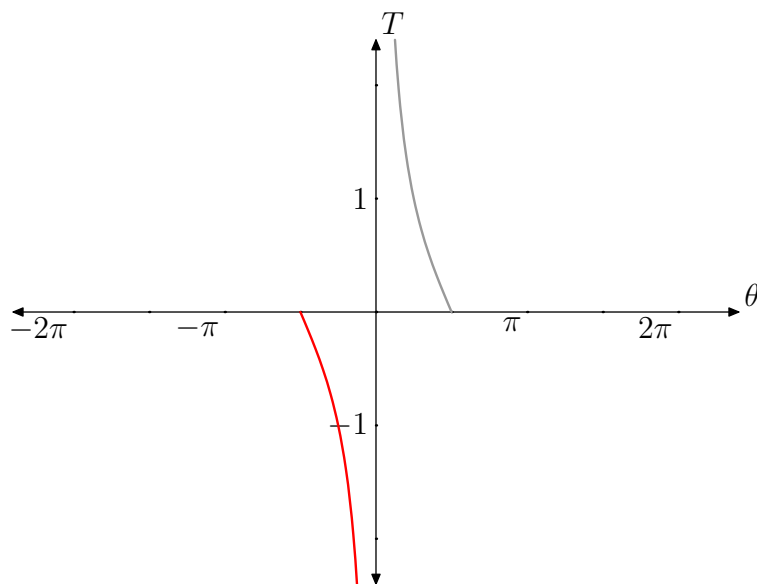
is small and positive and the numerator is near 1. As a consequence, $\cot(\theta)$ will be very large for small positive values of θ . This behavior will also create a vertical asymptote along the T -axis. As θ moves up to the value $\pi/2$, the denominator \sin will go to the value 1 while the numerator \cos will approach the value 0. Hence the ratio of these two quantities will tend to 0 as θ grows to $\pi/2$. All this provides the following picture of the graph of \cot for $0 < \theta \leq \pi/2$.



The rest of the graph can be deduced from this limited picture by applying various identities. First of all,

$$\cot(-\theta) = -\cot(\theta)$$

for all angles θ . Hence the graph is symmetric about the origin. This determines the graph for angles $-\pi/2 \leq \theta < 0$:

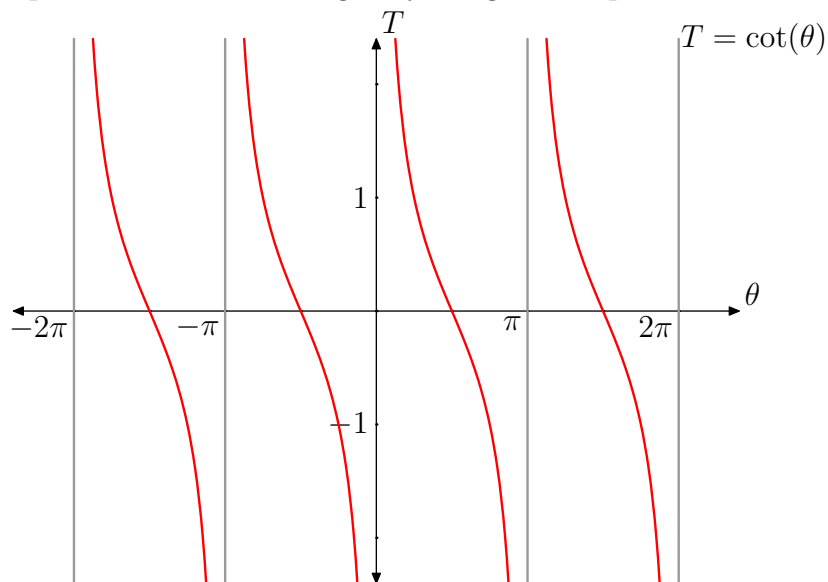


The rest of the graph can be obtained by applying the identity

$$\cot(\theta + \pi) = \cot(\theta)$$

CHAPTER 8. GRAPHS

for all angles θ . This implies that the graph of \cot is periodic with period π . The complete graph of \cot can be generated by shifting the above piece to the left and right by integer multiples of π .



Note that the vertical asymptotes occur at the places where $\sin(\theta)$ is zero.

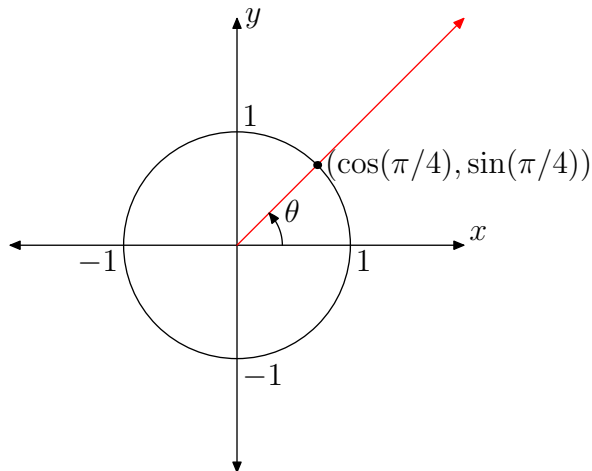
CHAPTER 9

Special Angles

It is not easy to calculate the exact values of the trigonometric functions for a general angle. One set of angles where such values are easy to calculate are integer multiples of $\pi/2$. These are angles for which, in the unit circle picture, the terminal ray points vertically up, vertically down, to the right, or to the left. This section treats three other angles for which the values of the trigonometric functions can be computed exactly. There are the angles $\theta = \pi/4, \pi/6$ and $\pi/3$.

1. The Angle $\pi/4$

The angle $\pi/4 = \pi/2 - \pi/4$ splits the first quadrant exactly in half. The terminal ray lies along the $y = x$ line.



So,

$$\cos(\pi/4) = \sin(\pi/4).$$

On the other hand, the Pythagorean theorem implies

$$\cos^2(\pi/4) + \sin^2(\pi/4) = 1.$$

Consequently,

$$2\cos^2(\pi/4) = 1$$

or

$$\cos^2(\pi/4) = \frac{1}{2}.$$

Taking square root yields,

$$\cos(\pi/4) = \pm \frac{1}{\sqrt{2}}.$$

Since, the terminal ray is in the first quadrant, $\cos(\pi/4)$ and $\sin(\pi/4)$ are both positive. In particular,

$$\cos(\pi/4) = \frac{1}{\sqrt{2}}$$

and hence

$$\sin(\pi/4) = \cos(\pi/4) = \frac{1}{\sqrt{2}}.$$

In summary,

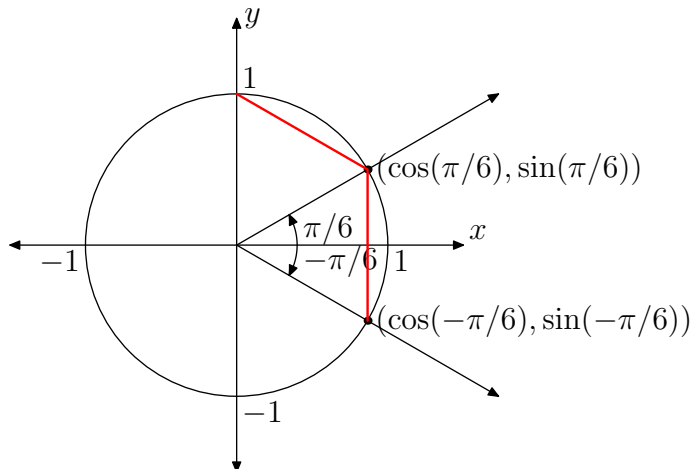
$$\cos(\pi/4) = \frac{1}{\sqrt{2}} \quad \text{and} \quad \sin(\pi/4) = \frac{1}{\sqrt{2}}.$$

2. The Angle $\pi/6$

To understand these special angles, first observe that

$$\frac{\pi}{3} = \frac{\pi}{2} - \frac{\pi}{6} \quad \text{and} \quad \frac{\pi}{3} = \frac{\pi}{6} - \left(-\frac{\pi}{6}\right).$$

With this in mind, it follows that the two line segments in the diagram below have the same length.



The lengths of each of these line segments can each be computed by using the distance formula. The endpoints for the topmost line segment are:

$$(0, 1) \quad \text{and} \quad (\cos(\pi/6), \sin(\pi/6))$$

and the endpoints for the vertical line segment are

$$(\cos(-\pi/6), \sin(-\pi/6)) \quad \text{and} \quad (\cos(\pi/6), \sin(\pi/6)).$$

By the distance formula:

$$\begin{aligned} \sqrt{(\cos(\pi/6) - 0)^2 + (\sin(\pi/6) - 1)^2} = \\ \sqrt{(\cos(\pi/6) - \cos(-\pi/6))^2 + (\sin(\pi/6) - \sin(-\pi/6))^2}. \end{aligned}$$

Recall the identities

$$\cos(-\theta) = \cos(\theta) \quad \text{and} \quad \sin(-\theta) = -\sin(\theta)$$

valid for all angles. Hence

$$\sqrt{\cos^2(\pi/6) + (\sin(\pi/6) - 1)^2} = \sqrt{(2\sin(\pi/6))^2}.$$

Upon squaring both sides and then expanding,

$$\cos^2(\pi/6) + \sin^2(\pi/6) - 2\sin(\pi/6) + 1 = 4\sin^2(\pi/6).$$

Applying the Pythagorean identities,

$$1 - 2\sin(\pi/6) + 1 = 4\sin^2(\pi/6)$$

or

$$0 = 4\sin^2(\pi/6) + 2\sin(\pi/6) - 2.$$

This leads to the quadratic equation

$$0 = 2\sin^2(\pi/6) + \sin(\pi/6) - 1.$$

The quadratic formula yields

$$\sin(\pi/6) = \frac{-1 \pm \sqrt{1^2 - 4(2)(-1)}}{4} = \frac{-1 \pm 3}{4} = \frac{1}{2}, -1$$

The angle $\pi/6$ is in the first quadrant and, hence, $\sin(\pi/6) > 0$. Thus,

$$\sin(\pi/6) = \frac{1}{2}.$$

By the Pythagorean identity,

$$\begin{aligned} 1 &= \cos^2(\pi/6) + \sin^2(\pi/6) \\ &= \cos^2(\pi/6) + \left(\frac{1}{2}\right)^2 \\ &= \cos^2(\pi/6) + \frac{1}{4}. \end{aligned}$$

Hence,

$$\cos^2(\pi/6) = \frac{3}{4}$$

or

$$\cos(\pi/6) = \pm \frac{\sqrt{3}}{2}.$$

Again, the fact that the terminal ray for $\pi/6$ is in the first quadrant means that we must choose the positive square root:

$$\cos(\pi/6) = \frac{\sqrt{3}}{2}.$$

In summary,

$$\cos(\pi/6) = \frac{\sqrt{3}}{2} \quad \text{and} \quad \sin(\pi/6) = \frac{1}{2}.$$

3. The Angle $\pi/3$

This case relies on the work already done with the angle $\pi/6$. Observe that,

$$\frac{\pi}{3} = \frac{\pi}{2} - \frac{\pi}{6}.$$

Identities relating an angle with its complementary angle have already been derived:

$$\cos(\pi/2 - \theta) = \sin(\theta) \quad \text{and} \quad \sin(\pi/2 - \theta) = \cos(\theta)$$

for all θ . Hence

$$\cos(\pi/3) = \cos(\pi/2 - \pi/6) = \sin(\pi/6) = \frac{1}{2}$$

and

$$\sin(\pi/3) = \sin(\pi/2 - \pi/6) = \cos(\pi/6) = \frac{\sqrt{3}}{2}.$$

In summary,

$$\cos(\pi/3) = \frac{1}{2} \quad \text{and} \quad \sin(\pi/3) = \frac{\sqrt{3}}{2}.$$

Question 3.1. Complete the following table:

θ	tan	cot	sec	csc
$\pi/4$				
$\pi/3$				
$\pi/6$				

CHAPTER 9. SPECIAL ANGLES

Solution:

By using the definitions for the trigonometric functions, it is easy to check that

θ	tan	cot	sec	csc
$\pi/4$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\pi/3$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
$\pi/6$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2

CHAPTER 10

Inverse Trigonometric Functions

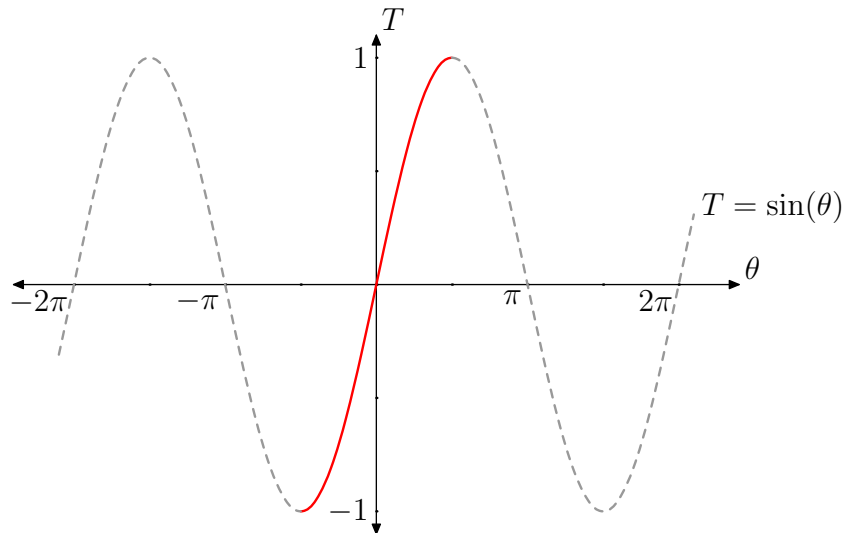
It is often necessary to calculate the angle that produces a given value for a trigonometric function. This can be done with the help of the *inverse trigonometric functions*. In this section, we define the functions \arcsin , \arccos and \arctan .

Recall that inverse functions are only defined when the original function is one-to-one, *i.e.* its graph satisfies the *horizontal* line test. The trigonometric functions are not one-to-one: in fact, there are horizontal lines that intersect the graph infinitely often.

To get around this difficulty, one restricts the domain so that the resulting graph is one-to-one. This restriction is called the *principal branch* of the trigonometric function.

1. Definition of \arcsin

The principal branch of the $T = \sin(\theta)$ is that part of the graph where $-\pi/2 \leq \theta \leq \pi/2$:

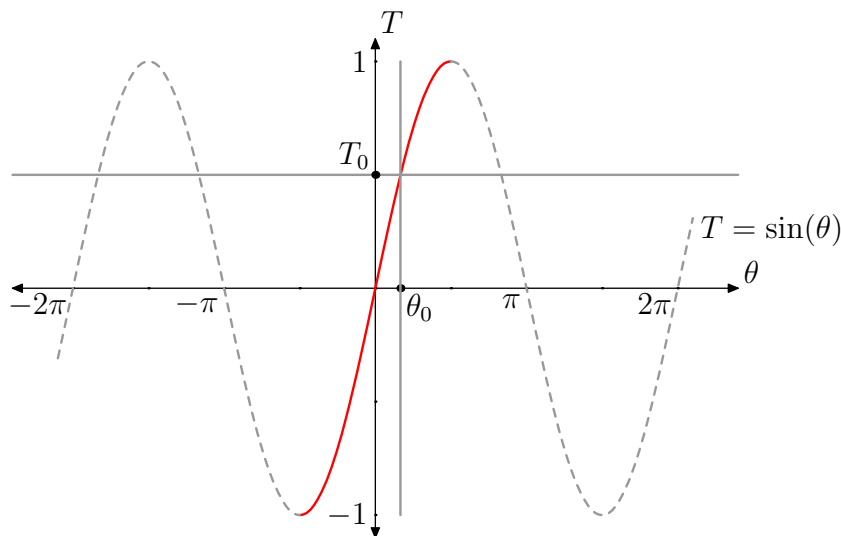


This principal branch takes on every possible value of the \sin function exactly once. For any T with $-1 \leq T \leq 1$, $\arcsin(T)$ is the unique

angle θ with $\pi/2 \leq \theta \leq \pi/2$ that has the property

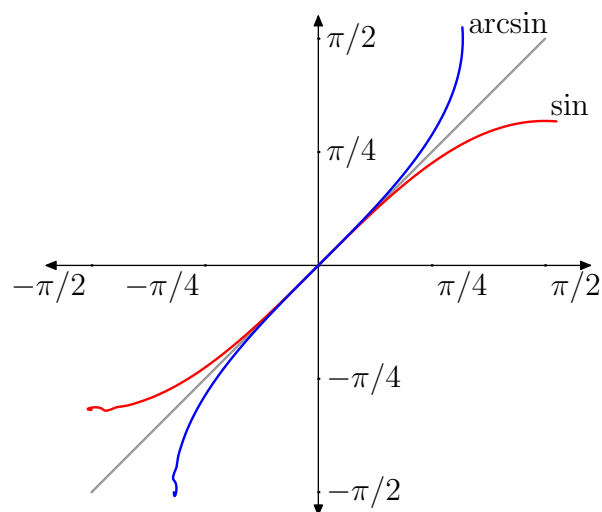
$$\sin(\theta) = T.$$

The definition is best understood graphically. Begin by drawing the horizontal line $T = T_0$ in the plot above.

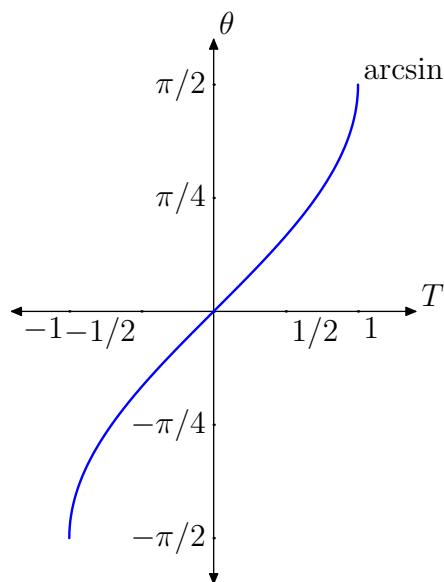


Although this horizontal line intersects the graph of \sin infinitely often, it intersects the principal branch exactly once. The θ -coordinate of this point is $\arcsin(T_0)$. In the plot it is labeled as θ_0 . Note that the point (θ_0, T_0) is on the graph \sin and the ordered pair (T_0, θ_0) is on the graph of \arcsin .

Hence, if one starts out with an ordered pair on the principal branch of \sin and reverses the coordinates, a point on the graph of \arcsin is obtained. This process works in reverse also: if (T_0, θ_0) is on the graph of \arcsin then (θ_0, T_0) is on the principal branch of \sin . Reversing the coordinates of a point reflects the point about the diagonal. So, the graph of \arcsin is obtained by reflecting the principal branch about the diagonal:



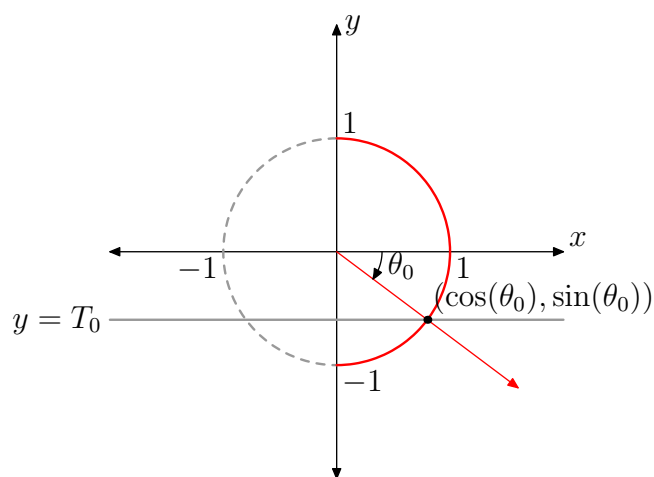
Reflecting the graph, interchanges the role of the domain and range. So, the domain of the arcsin function is the range of sin: $-1 \leq T \leq 1$. The range of arcsin is the domain of the principal branch of sin: $-\pi/2 \leq \theta \leq \pi/2$. Putting all this information together produces the following detailed plot of arcsin. !



The connection of all this with the unit circle picture is yet another important facet. Given a number T_0 with $-1 \leq T_0 \leq 1$, $\theta_0 = \arcsin(T_0)$ is the unique angle between $-\pi/2$ and $\pi/2$ with the property

$$\sin(\theta_0) = T_0.$$

How does one use the unit circle picture to graphically locate this angle?



Draw the horizontal line $y = T_0$ and locate its point of intersection with the right half of the unit circle. This point is on the terminal ray for an angle θ_0 where $-\pi/2 \leq \theta_0 \leq \pi/2$. Note that

$$\sin(\theta_0) = T_0.$$

Hence,

$$\theta_0 = \arcsin(T_0).$$

Question 1.1. Complete the following table:

T	$\theta_0 = \arcsin(T_0)$
0	
1/2	
$1/\sqrt{2}$	
$\sqrt{3}/2$	
-1/2	
$-1/\sqrt{2}$	
$-\sqrt{3}/2$	

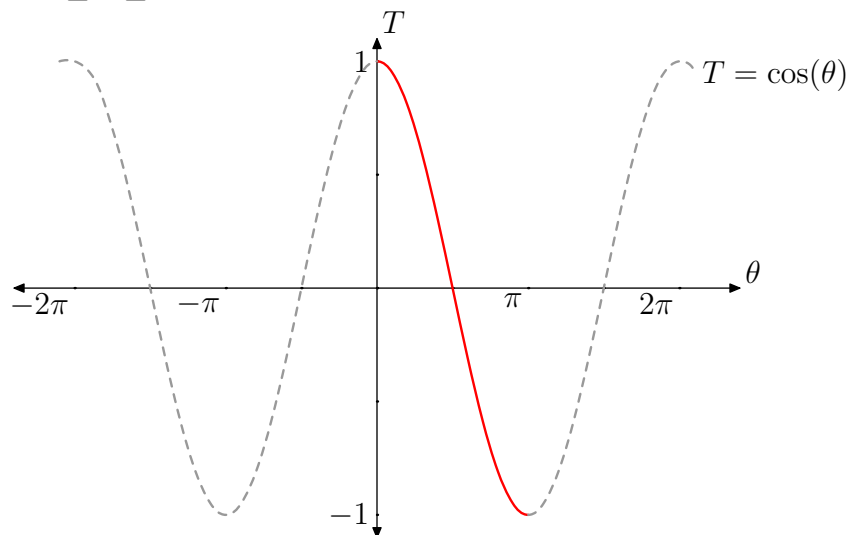
CHAPTER 10. INVERSE TRIGONOMETRIC FUNCTIONS

Solution:

T	$\theta_0 = \arcsin(T_0)$
0	0
1/2	$\pi/6$
$1/\sqrt{2}$	$\pi/4$
$\sqrt{3}/2$	$\pi/3$
-1/2	$-\pi/6$
$-1/\sqrt{2}$	$-\pi/4$
$-\sqrt{3}/2$	$-\pi/3$

2. Definition of \arccos

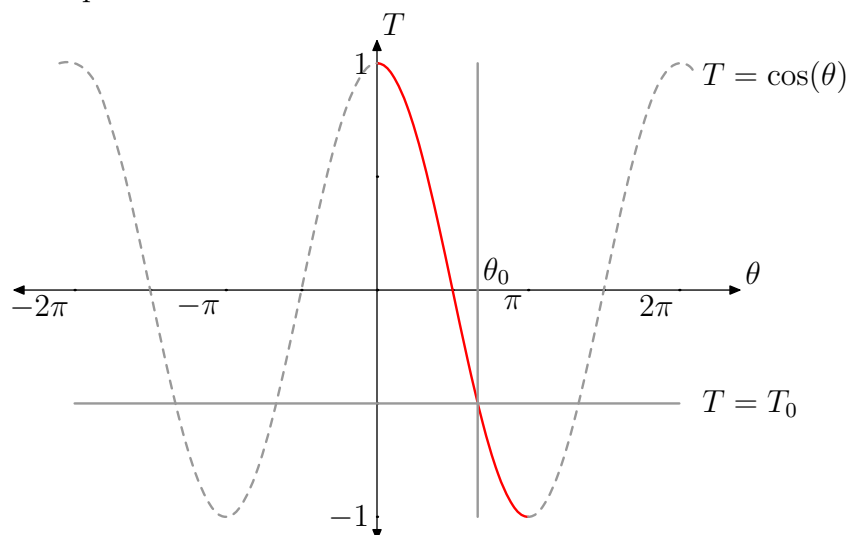
The principal branch of the $T = \cos(\theta)$ is that part of the graph where $0 \leq \theta \leq \pi$:



As was the case with \sin , the principal branch takes on every value of \cos exactly once. For any T with $-1 \leq T \leq 1$, $\arccos(T)$ is the unique angle θ with $0 \leq \theta \leq \pi$ that has the property

$$\cos(\theta) = T.$$

To further understand this definition, draw the horizontal line $T = T_0$ in the plot above.

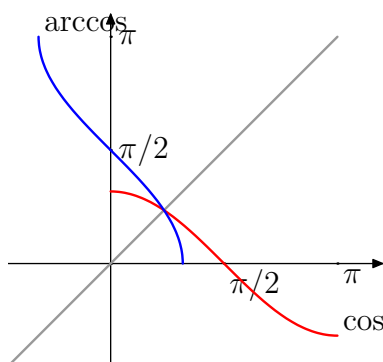


Although this horizontal line intersects the graph of \sin infinitely often, it intersects the principal branch exactly once. The θ -coordinate

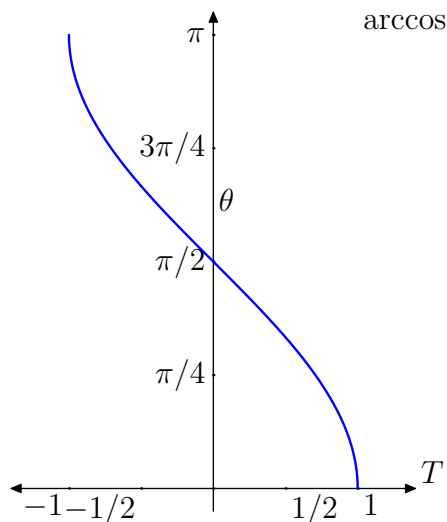
CHAPTER 10. INVERSE TRIGONOMETRIC FUNCTIONS

of this point is $\arccos(T_0)$. In the plot it is labeled as θ_0 . Note that the point (θ_0, T_0) is on the graph \cos and the ordered pair (T_0, θ_0) is on the graph of \arccos .

Hence, if one starts out with an ordered pair on the principal branch of \cos and reverses the coordinates, a point on the graph of \arccos is obtained. Thus, in analogy with the \arcsin function, the graph of \arccos is obtained by reflecting the principal branch of \cos about the diagonal:



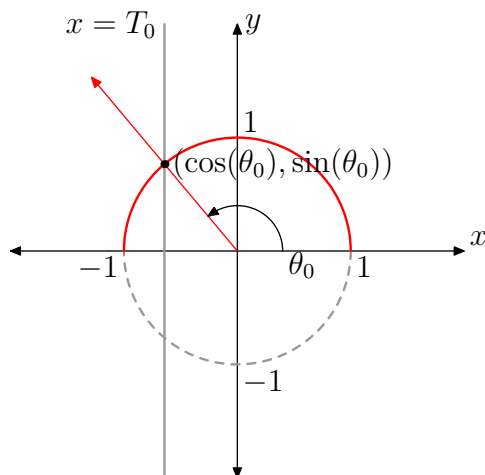
Reflecting the graph, interchanges the role of the domain and range. So, the domain of the \arccos function is the range of \sin : $-1 \leq T \leq 1$. The range of \arccos is the domain of the principal branch of \cos : $0 \leq \theta \leq \pi$. Putting all this information together produces the following detailed plot of \arccos :



The relationship to the unit circle picture is also important. Given a number T_0 with $-1 \leq T_0 \leq 1$, $\theta_0 = \arccos(T_0)$ is the unique angle between 0 and π with the property

$$\cos(\theta_0) = T_0.$$

How does one use the unit circle picture to graphically locate this angle?



Draw the vertical line $x = T_0$ and locate its point of intersection with the upper half of the unit circle. This point is on the terminal ray for an angle θ_0 where $0 \leq \theta_0 \leq \pi$. Note that

$$\cos(\theta_0) = T_0.$$

Hence,

$$\theta_0 = \arccos(T_0).$$

Question 2.1. Complete the following table:

T	$\theta_0 = \arccos(T_0)$
0	
$1/2$	
$1/\sqrt{2}$	
$\sqrt{3}/2$	
1	
$-1/2$	
$-1/\sqrt{2}$	
$-\sqrt{3}/2$	
-1	

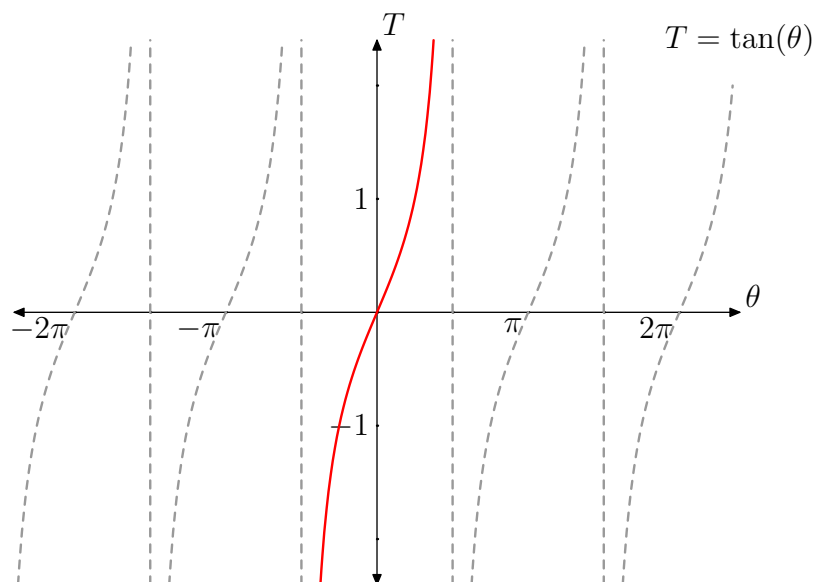
CHAPTER 10. INVERSE TRIGONOMETRIC FUNCTIONS

Solution:

T	$\theta_0 = \arccos(T_0)$
0	$\pi/2$
$1/2$	$\pi/3$
$1/\sqrt{2}$	$\pi/4$
$\sqrt{3}/2$	$\pi/6$
1	0
$-1/2$	$2\pi/3$
$-1/\sqrt{2}$	$3\pi/4$
$-\sqrt{3}/2$	$5\pi/6$
-1	π

3. Definition of \arctan

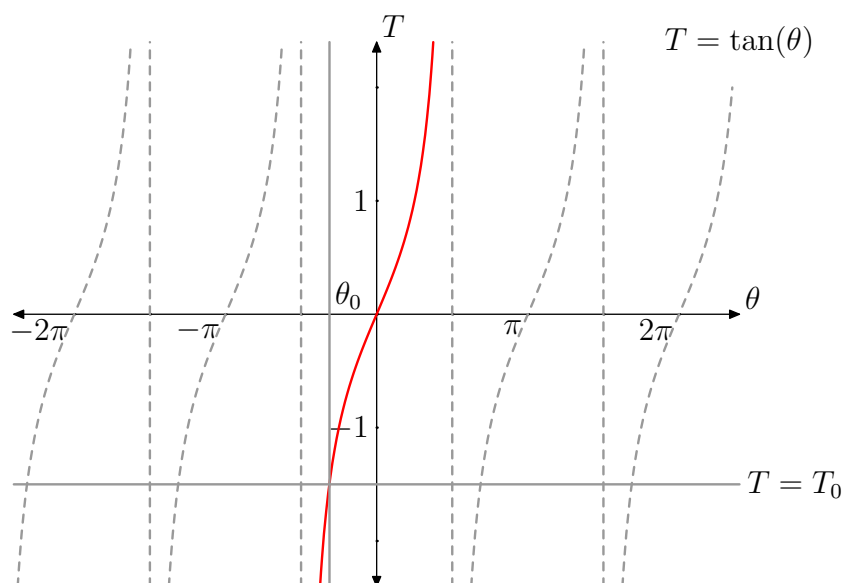
The principal branch of the $T = \tan(\theta)$ is that part of the graph where $-\pi/2 < \theta < \pi/2$:



Observe that the principal branch of \tan takes on every possible value exactly once. Given any number T , $\arctan(T)$ is defined to be the unique angle θ with $-\pi/2 < \theta < \pi/2$ and

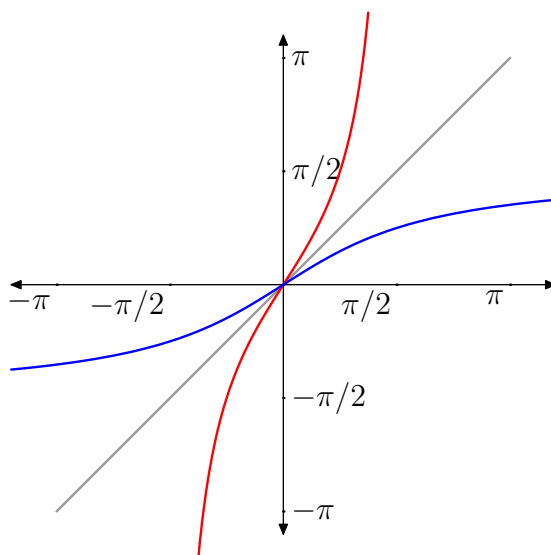
$$\tan(\theta) = T.$$

How does one graphically compute $\arctan(T_0)$ for some number T_0 ?



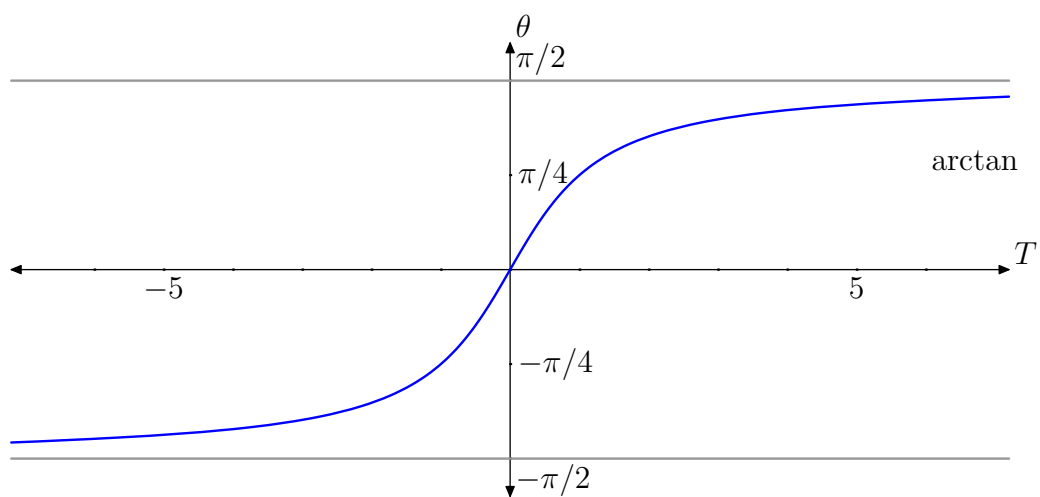
CHAPTER 10. INVERSE TRIGONOMETRIC FUNCTIONS

First, plot the horizontal line $T = T_0$ and note that this line intersects the principal branch at exactly one place. The first coordinate of this point of intersection, θ_0 is $\arctan(T_0)$. Observe that the ordered pair (T_0, θ_0) is on the graph of \arctan precisely when (θ_0, T_0) is on the principal branch of the \tan function. Thus, the graph of \arctan can be obtained from the principal branch by reflection about the diagonal.



Reflecting the graph, interchanges the role of the domain and range. So, the domain of the \arctan function is the range of \tan : all numbers T . The range of \arctan is the domain of the principal branch of \tan : $-\pi/2 < \theta < \pi/2$. Also important are the two vertical asymptotes of the principal branch of \tan . After reflection about the diagonal, these yield horizontal asymptotes to \arctan . Putting all this information together produces the following detailed plot of \arctan :





CHAPTER 10. INVERSE TRIGONOMETRIC FUNCTIONS

Question 3.1. Complete the following table:

T	$\theta_0 = \arctan(T_0)$
0	
$1/\sqrt{3}$	
1	
$\sqrt{3}$	
$-1/\sqrt{3}$	
-1	
$-\sqrt{3}$	

CHAPTER 10. INVERSE TRIGONOMETRIC FUNCTIONS

Solution:

T	$\theta_0 = \arctan(T_0)$
0	0
$1/\sqrt{3}$	$\pi/6$
1	$\pi/4$
$\sqrt{3}$	$\pi/3$
$-1/\sqrt{3}$	$-\pi/6$
-1	$-\pi/4$
$-\sqrt{3}$	$-\pi/3$

Part 3

Triangles

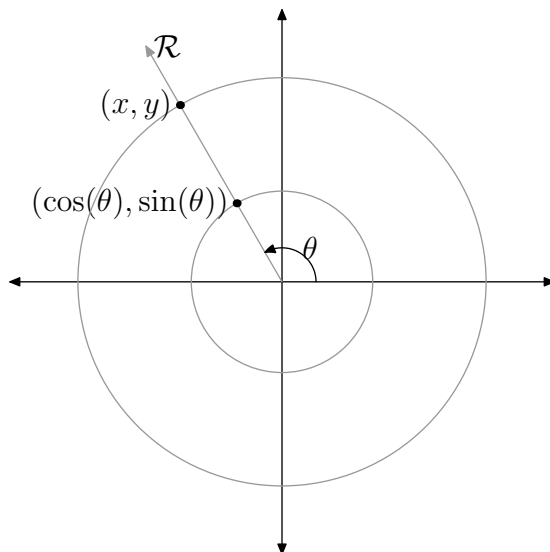
CHAPTER 11

Polar Co-ordinates

The typical method of describing the location of a point is by means of its Cartesian co-ordinates (x, y) . A second method, using trigonometry, is often useful. *Polar co-ordinates* locate a point by giving its distance to the origin and the ray \mathcal{R} (with vertex at the origin) on which the point lies. The distance of the point (x, y) to the origin, using the distance formula, is

$$r = \sqrt{(x - 0)^2 + (y - 0)^2} = \sqrt{x^2 + y^2}.$$

The ray \mathcal{R} described above is the terminal ray of some standard angle with radian measure θ . The pair of numbers r and θ are the polar co-ordinates of the point.



How are the Cartesian co-ordinates (x, y) related to the polar co-ordinates r, θ ? Observe that the ray \mathcal{R} intersects the circle with radius r at the point (x, y) and the unit circle at the point $(\cos(\theta), \sin(\theta))$. Magnify the unit circle by a factor of r and one gets the second circle. It follows that x and y are obtained from $\cos(\theta)$ and $\sin(\theta)$ by multiplication by r :

CHAPTER 11. POLAR CO-ORDINATES

$$x = r \cos(\theta)$$

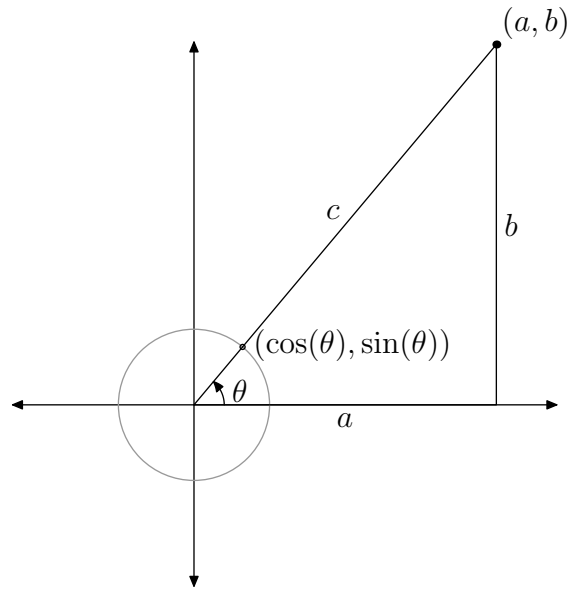
$$y = r \sin(\theta)$$

CHAPTER 12

Right Triangles

The same idea described in the last section can be applied to deduce a very important connection between right triangles and the trigonometric functions.

Consider a right triangle with hypotenuse of length c and two legs of length a and b . Draw co-ordinate axes together with the unit circle, as shown.



By reasoning as in the previous section,

$$a = c \cos(\theta)$$

$$b = c \sin(\theta).$$

(Here the length c plays the same role as the distance r .) Dividing both sides of these equations by c yields:

$$\cos(\theta) = \frac{a}{c}$$

$$\sin(\theta) = \frac{b}{c}.$$

! .
Label c by hyp (for hypotenuse), a by adj since it is adjacent to the central angle and b by opp since it is opposite the central angle. The preceding set of equations become

$$\begin{aligned}\cos(\theta) &= \frac{\text{adj}}{\text{hyp}} \\ \sin(\theta) &= \frac{\text{opp}}{\text{hyp}}.\end{aligned}$$

Question 0.2. Find similar formulae between the other four trigonometric functions and the sides of a right triangle.

Solution:

$$\tan(\theta) = \frac{\frac{\text{opp}}{\text{hyp}}}{\frac{\text{adj}}{\text{hyp}}} = \frac{\text{opp}}{\text{adj}}$$

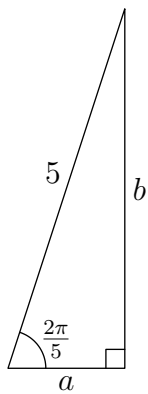
$$\cot(\theta) = \frac{1}{\tan(\theta)} = \frac{\text{adj}}{\text{opp}}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)} = \frac{\text{hyp}}{\text{adj}}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)} = \frac{\text{hyp}}{\text{opp}}$$

CHAPTER 12. RIGHT TRIANGLES

Question 0.3. Find the unknown sides and angles in the right triangle shown below.



CHAPTER 12. RIGHT TRIANGLES

Solution: The cosine of an acute angle within a right triangle is opposite divided by hypotenuse.

$$\cos(2\pi/5) = \frac{a}{5}$$

$$a = 5 \cos(2\pi/5) \approx 1.545$$

The sine of an acute angle within a right triangle is adjacent divided by hypotenuse.

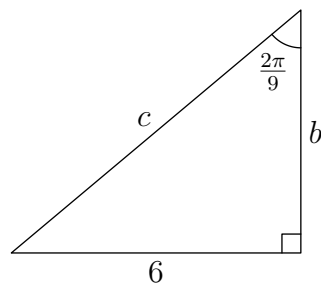
$$\sin(2\pi/5) = \frac{b}{5}$$

$$b = 5 \sin(2\pi/5) \approx 4.755$$

Finally, since the sum of the angles in a triangle is π radians, the third angle must be

$$\pi - \left(\frac{\pi}{2} + \frac{2\pi}{5}\right) = \frac{\pi}{10}.$$

Question 0.4. Find the unknown sides and angles in the right triangle shown below.



CHAPTER 12. RIGHT TRIANGLES

Solution: The tangent of an acute angle in a right triangle is opposite divided by adjacent. So,

$$\begin{aligned}\tan\left(\frac{2\pi}{9}\right) &= 6b \\ b &= \frac{6}{\tan\left(\frac{2\pi}{9}\right)} \approx 7.151\end{aligned}$$

The length of the hypotenuse is

$$\begin{aligned}\sin\left(\frac{2\pi}{9}\right) &= \frac{6}{c} \\ c &= \frac{6}{\sin\left(\frac{2\pi}{9}\right)} \approx 9.33\end{aligned}$$

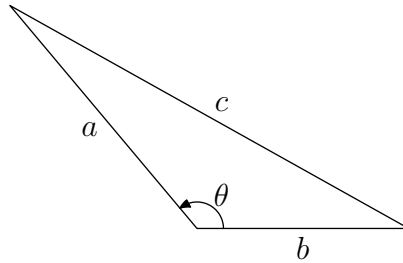
The third angle is

$$\pi - \left(\frac{\pi}{2} + \frac{2\pi}{9}\right) = \frac{5\pi}{18}$$

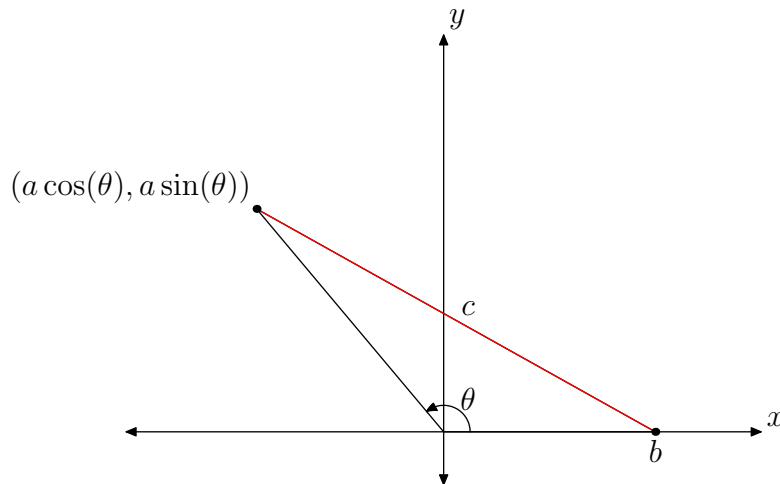
CHAPTER 13

Law of Cosines

Polar co-ordinates are also at the root of a relationship between the angles and sides of a general triangle. Begin with such a triangle:



Draw co-ordinates onto this picture so that the origin is at the vertex of the angle θ and the x -axis is along the side with length b :



On the one hand the length of the red segment is c , and on the other it is the distance between the points $(b, 0)$ and $(a \cos(\theta), a \sin(\theta))$. This observation leads to the equation:

$$c = \sqrt{(a \cos(\theta) - b)^2 + (a \sin(\theta) - 0)^2}.$$

Square both sides and expanding the right hand sides.

$$c^2 = a^2 \cos^2(\theta) - 2ab \cos(\theta) + b^2 + a^2 \sin^2(\theta).$$

Collect the two terms involving a^2 and rearrange.

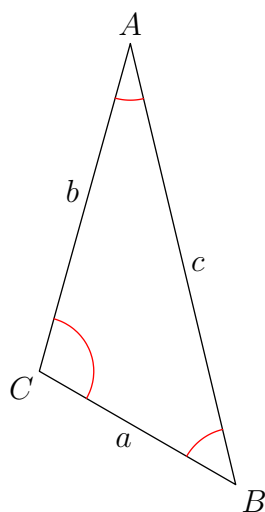
$$c^2 = a^2(\cos^2(\theta) + \sin^2(\theta)) + b^2 - 2ab \cos(\theta).$$

Finally, applying the Pythagorean identity,

$$c^2 = a^2 + b^2 - 2ab \cos(\theta).$$

This is the *Law of Cosines*.

Question 0.5 (Side-Side-Side (SSS)). The three sides of the triangle below are $a = 2$, $b = 3$ and $c = 4$. Find the three angles.



CHAPTER 13. LAW OF COSINES

Solution: The Law of Cosines can be applied to the triangle three different ways:

$$c^2 = a^2 + b^2 - 2ab \cos(\angle C)$$

$$b^2 = a^2 + c^2 - 2ac \cos(\angle B)$$

$$a^2 = b^2 + c^2 - 2bc \cos(\angle A)$$

The first of these yields:

$$4^2 = 2^2 + 3^2 - 2 \cdot 2 \cdot 3 \cdot \cos(\angle C)$$

$$12 \cos(\angle C) = -3$$

$$\cos(\angle C) = -0.25$$

$$\angle C = \arccos(-.25) \approx 1.823 \text{ radians}$$

The second yields:

$$3^2 = 2^2 + 4^2 - 2 \cdot 2 \cdot 4 \cdot \cos(\angle B)$$

$$\cos(\angle B) = \frac{11}{16}$$

$$\angle B = \arccos\left(\frac{11}{16}\right) \approx .8128 \text{ radians}$$

The third angle can be found by the same method. However, it is easier to use the fact that the three angles in a triangle sum to π radians.

$$\angle A + \angle B + \angle C = \pi$$

$$\angle A = \pi - (\angle B + \angle C) \approx .5058 \text{ radians}$$

The shortest distance between two points is achieved by the line segment connecting them. It follows that the length of any leg in a triangle must be smaller than the sum of the lengths of the other two legs. If this *triangle inequality* is violated in an SSS problem, there is no solution.

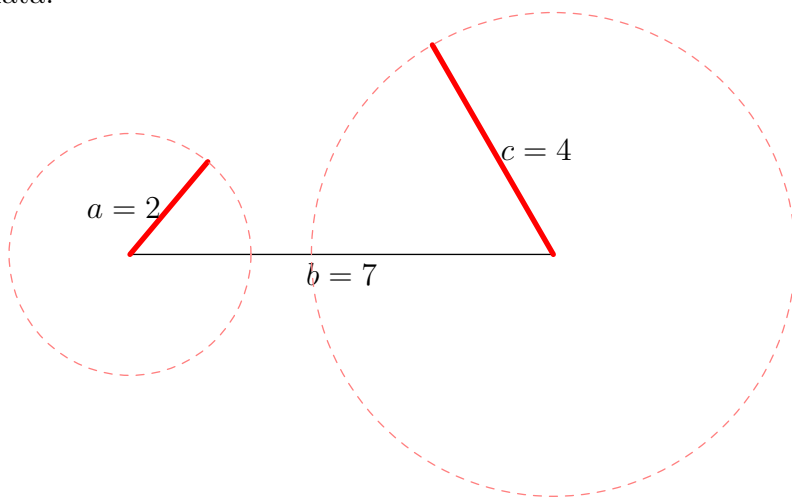
Question 0.6 (SSS (no solution)). The three sides of a triangle are $a = 2, b = 7$ and $c = 4$. Find the three angles.

CHAPTER 13. LAW OF COSINES

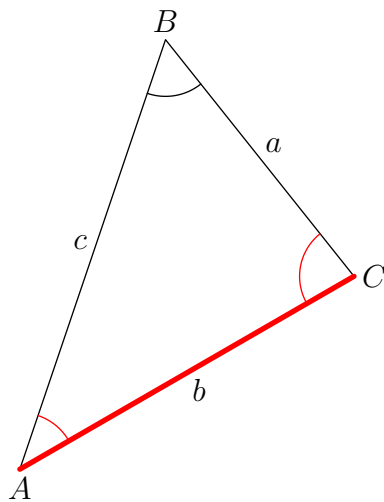
Solution: Observe that

$$a + c = 6 < 7 = b$$

The triangle inequality is violated and hence there is no triangle with this data.



Question 0.7 (Side-Angle-Side (SAS)). In the triangle below $a = 4$, $c = 6$ and $\angle B = 1$ rad. Find the remaining leg and two angles.



*Solution:*Start with

$$b^2 = a^2 + c^2 - 2ac \cos(\angle B).$$

Substitute the known values and solve for $\cos(\angle B)$:

$$b^2 = 4^2 + 6^2 - 48 \cos(1)$$

$$b^2 \approx 26.065$$

$$b \approx 5.105$$

Next, solve for the angle $\angle C$:

$$c^2 = a^2 + b^2 - 2ab \cos(\angle C)$$

$$6^2 \approx 4^2 + 5.105^2 - 2 \cdot 4 \cdot 5.105 \cdot \cos(\angle C)$$

$$\cos(\angle C) \approx .1484$$

$$\angle C \approx 1.422$$

The final angle is calculated by

$$\angle A \approx \pi - (1 + 1.422) \approx 0.7186$$

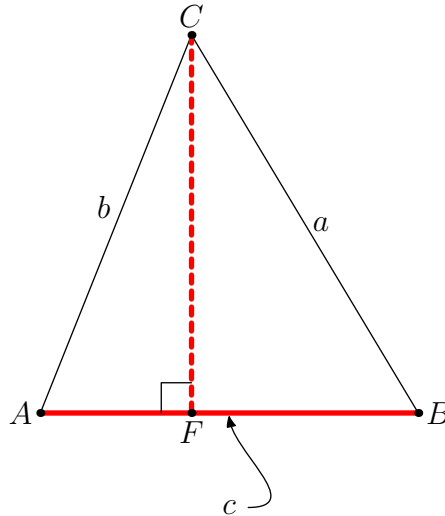
CHAPTER 14

Law of Sines

The area of a triangle is

$$\frac{1}{2} \times \text{base} \times \text{altitude}.$$

In the triangle below the base is c and the altitude is the line segment CF . Label the length of CF by h .



The line segment CF is a leg in the right triangle ACF . The hypotenuse is the segment AC with length b and CF is opposite the angle $\angle A$. So

$$\sin(\angle A) = \frac{h}{b}.$$

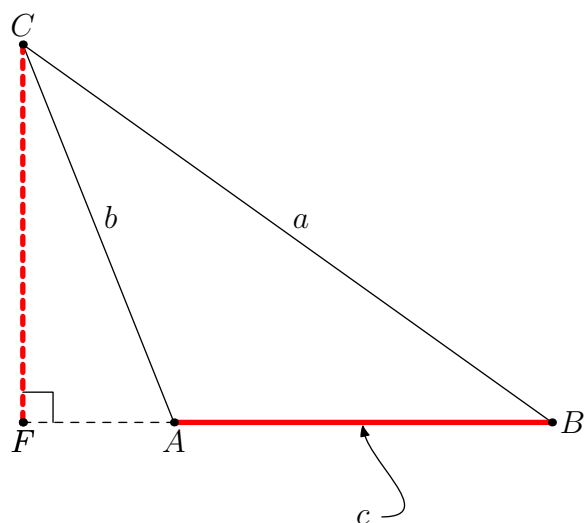
Multiplying both sides by b yields

$$h = b \sin(\angle A).$$

Hence the area of the triangle is

$$\text{Area} = \frac{1}{2} cb \sin(\angle A).$$

However, the picture need not look like the one above: the foot of the altitude F may not be in between A and B . Such a situation is illustrated below.



Does the same principle for computing the area still apply? In this situation,

$$\sin(\angle FAC) = \frac{h}{b}.$$

Solving h yields,

$$h = b \sin(\angle FAC) = \sin(\pi - \angle CAB).$$

Since supplementary angles have the same sine,

$$\sin(\pi - \angle CAB) = \sin(\angle CAB) = \sin(\angle A).$$

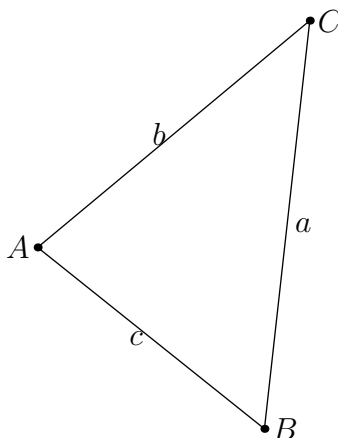
Therefore, once again,

$$\text{Area} = \frac{1}{2}cb \sin(\angle A).$$

In words: *the area of a triangle is equal to one half the product of two sides times the sine of the included angle.*

The above principle applies to a general triangle three different ways:

$$\text{Area} = \frac{1}{2}cb \sin(\angle A) = \frac{1}{2}ab \sin(\angle C) = \frac{1}{2}ac \sin(\angle B)$$



In particular,

$$\frac{1}{2}cb \sin(\angle A) = \frac{1}{2}ab \sin(\angle C).$$

Canceling common factors,

$$c \sin(\angle A) = a \sin(\angle C).$$

Dividing both sides by ac , yields

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle C)}{c}.$$

The same process applied to

$$\frac{1}{2}ab \sin(\angle C) = \frac{1}{2}ac \sin(\angle B)$$

yields

$$\frac{\sin(\angle C)}{c} = \frac{\sin(\angle B)}{b}.$$

The *Law of Sines* is obtained by combining these observations,

$$\frac{\sin(\angle A)}{a} = \frac{\sin(\angle B)}{b} = \frac{\sin(\angle C)}{c}.$$

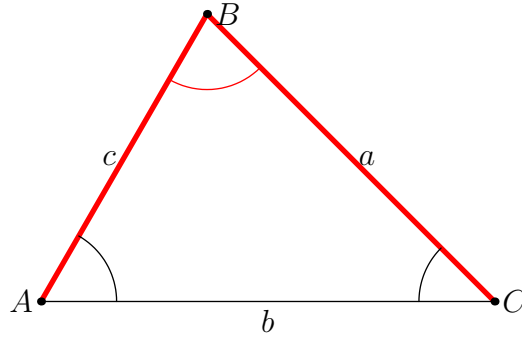
Note that is the above three fractions are equal their reciprocals must also be equal:

$$\frac{a}{\sin(\angle A)} = \frac{b}{\sin(\angle B)} = \frac{c}{\sin(\angle C)}.$$

A reformulation that is often convenient.



Question 0.8 (Angle-Side-Angle (ASA)). In the triangle shown below, $b = 3$, $\angle A = \pi/3$ radians and $\angle C = \pi/4$ radians. Find the remaining angle and two sides.



Solution: The third angle is

$$\begin{aligned}\angle B &= \pi - (\angle A + \angle C) \\ &= \pi - (\pi/3 + \pi/4) \\ &= \pi - \frac{7\pi}{12} \\ &= \frac{5\pi}{12}\end{aligned}$$

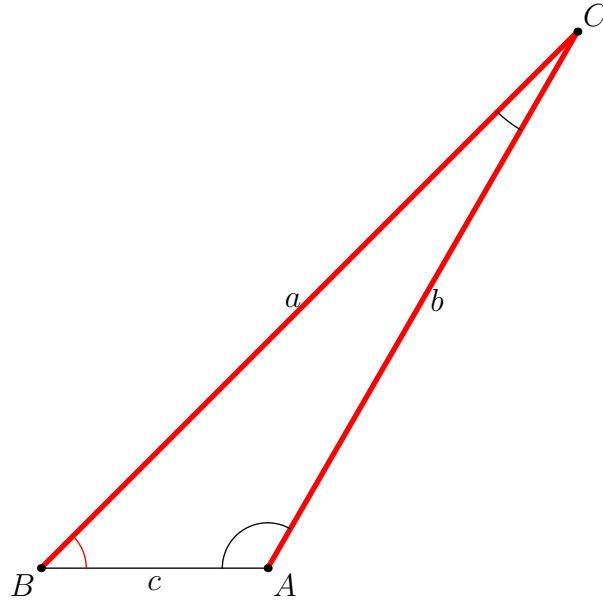
The two sides are calculated using the Law of Sines. First the side c :

$$\begin{aligned}\frac{c}{\sin(\angle C)} &= \frac{b}{\sin(\angle B)} \\ c &= \sin(\angle C) \frac{b}{\sin(\angle B)} \\ c &= \sin(\pi/4) \frac{3}{\sin(\frac{5\pi}{12})} \\ c &\approx 2.196\end{aligned}$$

The side a is calculated similarly:

$$\begin{aligned}a &= \sin(\angle A) \frac{b}{\sin(\angle B)} \\ a &= \sin(\pi/3) \frac{3}{\sin(\frac{5\pi}{12})} \\ a &\approx 2.690\end{aligned}$$

Question 0.9 (Side-Angle-Angle (SAA)). In the triangle shown below, $c = 10$, $\angle A = 2\pi/3$ radians and $\angle C = \pi/12$ radians. Find the remaining angle and two sides.



*Solution:*Begin by solving for the third angle:

$$\begin{aligned}\angle B &= \pi - (\angle A + \angle C) \\ &= \pi - (2\pi/3 + \pi/12) \\ &= \pi - \frac{3\pi}{4} = \pi/4.\end{aligned}$$

The Law of Sines can now be applied to solve for the remaining sides:

$$\begin{aligned}b &= \sin(\angle B) \frac{c}{\sin(\angle C)} \\ b &= \sin(\pi/4) \frac{10}{\sin(\frac{\pi}{12})} \\ b &\approx 27.32\end{aligned}$$

$$\begin{aligned}a &= \sin(\angle A) \frac{c}{\sin(\angle C)} \\ a &= \sin(2\pi/3) \frac{10}{\sin(\pi/12)} \\ a &\approx 33.46\end{aligned}$$

The final class of problems involving triangles in this chapter is the Angle-Side-Side problem. This type is complicated by the fact that depending on the given data there can be no solutions, one solution or two solutions. The final four examples treat each one of these cases.

Question 0.10 (Angle-Side-Side (ASS), No Solution Case). Using the usual notation for triangles, $\angle B = \pi/6$, $a = 2$ and $b = 6$. Find the remaining two angles and side.

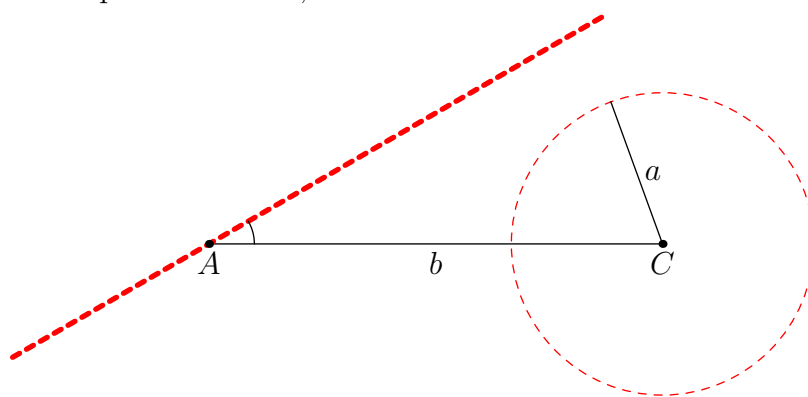
*Solution:*Begin with

$$\sin(\angle B) = b \frac{\sin(\angle A)}{a}.$$

Substituting known values yields the equation

$$\sin(\angle B) = 6 \frac{\sin(\pi/6)}{2} = \frac{3}{2} = 1.5.$$

Recall that the values of the sine function lie between the numbers -1 and 1 , inclusive. It follows that there can be no angle $\angle B$ that satisfies the above equation. Hence, no solution.



Question 0.11 (Angle-Side-Side (ASS), Unique Solution (Right Triangle)). Using the usual notation for triangles, $\angle B = \pi/6$, $a = 3$ and $b = 6$. Find the remaining two angles and side.

CHAPTER 14. LAW OF SINES

Solution: Once again, begin with

$$\sin(\angle B) = b \frac{\sin(\angle A)}{a}.$$

Substituting known values yields the equation

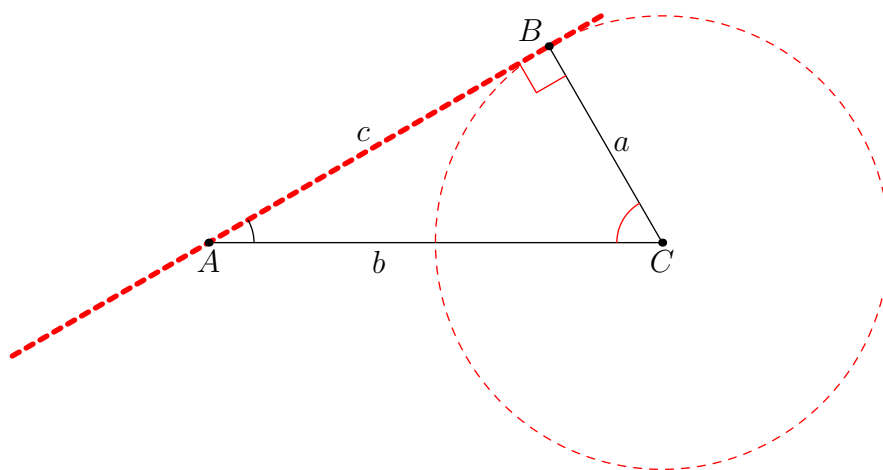
$$\sin(\angle B) = 6 \frac{\sin(\pi/6)}{3} = 1.$$

Since $\angle B$ is an unoriented angle within a triangle, it must be between 0 and π radians inclusive. The only angle in this interval whose sine is one is $\angle B = \pi/2$ radians. The third angle is

$$\angle C = \pi - (\angle A + \angle B) = \pi - (\pi/6 + \pi/2) = \pi/3.$$

A final application of the Law of Sines gives

$$\begin{aligned} c &= \sin(\angle C) \frac{a}{\sin(\angle A)} \\ &= \sin(\pi/3) \frac{3}{\sin(\pi/6)} \\ &= \frac{\sqrt{3}}{2} \frac{3}{1/2} = 3\sqrt{3}. \end{aligned}$$



Question 0.12 (Angle-Side-Side (ASS), Two Solutions). Using the usual notation for triangles, $\angle B = \pi/6$, $a = 2\sqrt{3}$ and $b = 6$. Find the remaining two angles and side.

*Solution:*Begin with

$$\sin(\angle B) = b \frac{\sin(\angle A)}{a}.$$

Substituting known values yields the equation

$$\sin(\angle B) = 6 \frac{\sin(\pi/6)}{2\sqrt{3}} = \frac{\sqrt{3}}{2}.$$

Since $\angle B$ is an angle in a triangle, any solution to the above equation between 0 and π radians must be considered. One possibility is

$$\angle B = \arcsin(\sqrt{3}/2) = \frac{\pi}{3}.$$

Since supplementary angles have the same sine, another choice is

$$\angle B = \pi - \arcsin(\sqrt{3}/2) = \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

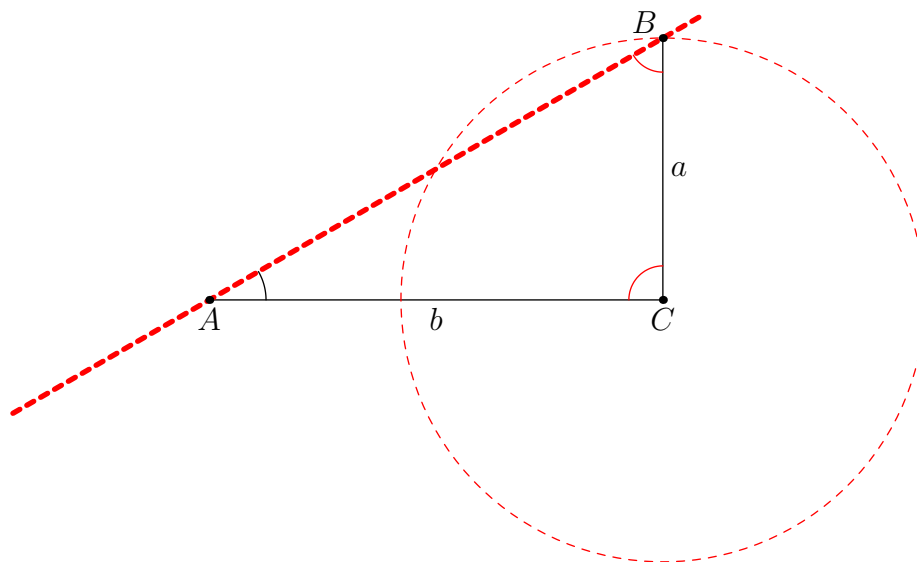
Case 1: $\angle B = \frac{\pi}{3}$

The third angle is

$$\angle C = \pi - (\angle A + \angle B) = \pi - (\pi/6 + \pi/3) = \pi/2.$$

Finally,

$$\begin{aligned} c &= \sin(\angle C) \frac{a}{\sin(\angle A)} \\ &= \sin(\pi/2) \frac{2\sqrt{3}}{\sin(\pi/6)} = 4\sqrt{3} \end{aligned}$$



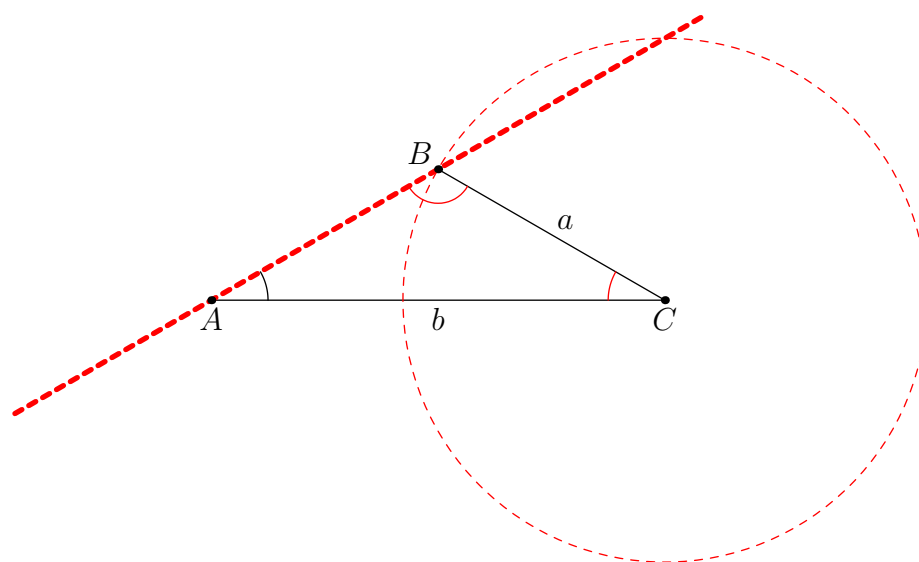
Case 2: $\angle B = \frac{2\pi}{3}$

In this case,

$$\angle C = \pi - (\angle A + \angle B) = \pi - (\pi/6 + 2\pi/3) = \pi/6.$$

Finally,

$$\begin{aligned} c &= \sin(\angle C) \frac{a}{\sin(\angle A)} \\ &= \sin(\pi/6) \frac{2\sqrt{3}}{\sin(\pi/6)} = 2\sqrt{3} \end{aligned}$$



CHAPTER 14. LAW OF SINES

Question 0.13 (Angle-Side-Side (ASS), One Solution). Using the usual notation for triangles, $\angle B = \pi/6$, $a = 7$ and $b = 6$. Find the remaining two angles and side.

Solution: Using the Law of Sines as before,

$$\begin{aligned}\sin(\angle B) &= b \frac{\sin(\angle A)}{a} \\ &= 6 \frac{\sin(\pi/6)}{7} \\ &= 3/7\end{aligned}$$

Since $\angle B$ is an angle in a triangle, any solution to the above equation between 0 and π radians must be considered. The solution between 0 and $\pi/2$ is

$$\angle B = \arcsin(3/7) \approx .4429.$$

The supplementary angle also has the correct sine:

$$\angle B = \pi - \arcsin(3/7) \approx 2.6987.$$

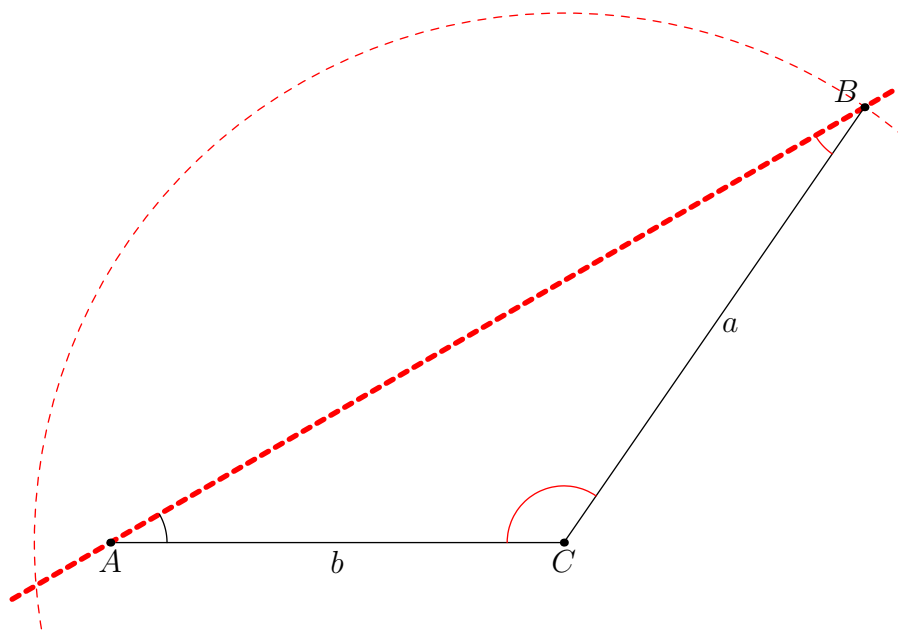
Case 1: $\angle B = \arcsin(3/7) \approx .4429$

The third angle is

$$\angle C = \pi - (\angle A + \angle B) = \pi - (\pi/6 + \arcsin(3/7)) \approx 2.1751.$$

The third leg is

$$\begin{aligned}c &= \sin(\angle C) \frac{a}{\sin(\angle A)} \\ &\approx \sin(2.1751) \frac{7}{1/2} = 4\sqrt{3} \\ &\approx 11.5207\end{aligned}$$



CHAPTER 14. LAW OF SINES

Case 2: $\angle B = \pi - \arcsin(3/7) \approx 2.6987$

In this case, the third angle would have to be

$$\begin{aligned}\angle C &= \pi - (\angle A + \angle B) \\ &\approx \pi - (\pi/6 + 2.6987) \\ &\approx -0.8071\end{aligned}$$

However, since the radian measure of an unoriented angle must be between 0 and π radians, this case must be discarded.

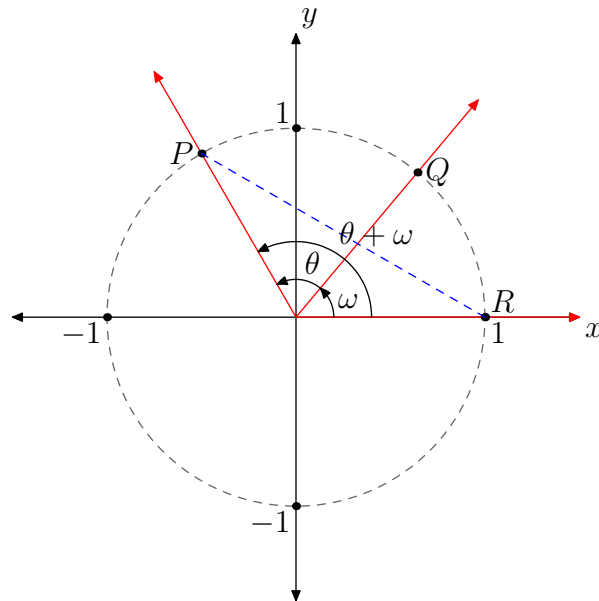
Part 4

Identities

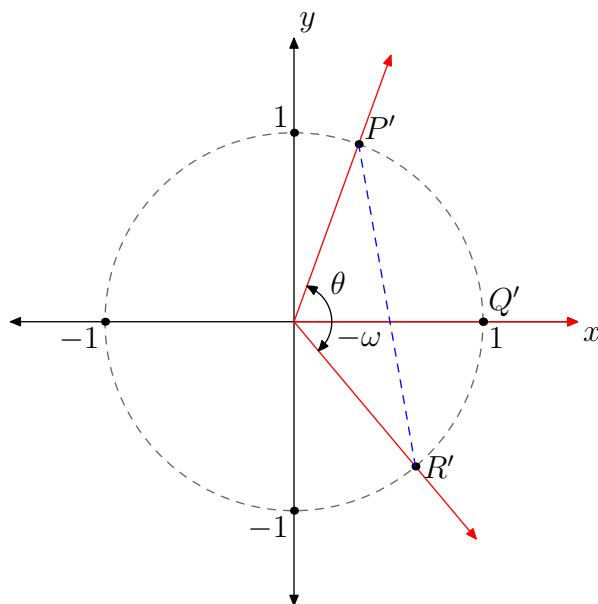
CHAPTER 15

Sum and Difference Formulas

Consider the diagram below where the angle ω is followed by an angle θ to produce a cumulative angle $\theta + \omega$. The co-ordinates of the point labeled R are $(1, 0)$ while the co-ordinates of the point P are $(\cos(\theta + \omega), \sin(\theta + \omega))$.



Next, rotate the three marked rays clockwise so the the middle ray lies along the positive x -axis. The points P , Q and R get rotated to P' , Q' and R' .



The co-ordinates of P' are $(\cos(\theta), \sin(\theta))$, while the co-ordinates of R' are $(\cos(-\omega), \sin(-\omega))$.

Rotations do not distort distances. Consequently, the length of the line segment PR is the same as the length of the line segment $P'R'$. Both of these distances can be computed by the distance formula.

$$\begin{aligned} \text{dist}(P, R) &= \sqrt{(\cos(\theta + \omega) - 1)^2 + (\sin(\theta + \omega) - 0)^2} \\ &= \sqrt{\cos^2(\theta + \omega) - 2\cos(\theta + \omega) + 1 + \sin^2(\theta + \omega)} \\ &= \sqrt{2 - 2\cos(\theta + \omega)} \end{aligned}$$

(Here we have used the Pythagorean identity: $\cos^2(A) + \sin^2(A) = 1$.)

Next:

$$\begin{aligned} \text{dist}(P', R') &= \sqrt{(\cos(\theta) - \cos(-\omega))^2 + (\sin(\theta) - \sin(-\omega))^2} \\ &= \sqrt{(\cos(\theta) - \cos(\omega))^2 + (\sin(\theta) + \sin(\omega))^2} \end{aligned}$$

The last step applies the fact that \cos is an even function and \sin is an odd function. To simplify further, expand each of the squared expressions underneath the last radical:

$$\begin{aligned} (\cos(\theta) - \cos(\omega))^2 &= \cos^2(\theta) - 2\cos(\theta)\cos(\omega) + \cos^2(\omega) \\ (\sin(\theta) + \sin(\omega))^2 &= \sin^2(\theta) + 2\sin(\theta)\sin(\omega) + \sin^2(\omega) \end{aligned}$$

Adding the two expressions and applying the Pythagorean identity twice yields:

$$(\cos(\theta) - \cos(\omega))^2 + (\sin(\theta) + \sin(\omega))^2 = 2 - 2\cos(\theta)\cos(\omega) + \sin(\theta)\sin(\omega).$$

CHAPTER 15. SUM AND DIFFERENCE FORMULAS

Substituting back into the previous expression for $\text{dist}(P', R')$ yields:

$$\text{dist}(P', R') = \sqrt{2 - 2 \cos(\theta) \cos(\omega) + \sin(\theta) \sin(\omega)}.$$

Since $\text{dist}(P, R) = \text{dist}(P', Q')$,

$$\sqrt{2 - 2 \cos(\theta + \omega)} = \sqrt{2 - 2 \cos(\theta) \cos(\omega) + \sin(\theta) \sin(\omega)}.$$

Square both sides and simplify as follows,

$$\begin{aligned} 2 - 2 \cos(\theta + \omega) &= 2 - 2 \cos(\theta) \cos(\omega) + \sin(\theta) \sin(\omega) \\ -2 \cos(\theta + \omega) &= -2 \cos(\theta) \cos(\omega) + \sin(\theta) \sin(\omega) \\ \cos(\theta + \omega) &= \cos(\theta) \cos(\omega) - \sin(\theta) \sin(\omega) \end{aligned}$$

This last formula is the sum formula for the cosine function. The difference formula is obtained as follows:

$$\begin{aligned} \cos(\theta - \omega) &= \cos(\theta - \omega) \\ &= \cos(\theta) \cos(-\omega) - \sin(\theta) \sin(-\omega) \\ &= \cos(\theta) \cos(\omega) + \sin(\theta) \sin(\omega) \end{aligned}$$

The analogous formulae for sine are obtained from those for cosine. Recall that the sine of an angle is equal to the cosine of the complementary angle.

$$\begin{aligned} \sin(\theta + \omega) &= \cos(\pi/2 - (\theta + \omega)) \\ &= \cos(\pi/2 - \theta - \omega) \\ &= \cos((\pi/2 - \theta) - \omega) \\ &= \cos(\pi/2 - \theta) \cos(\omega) + \sin(\pi/2 - \theta) \sin(\omega) \\ &= \sin(\theta) \cos(\omega) + \cos(\theta) \sin(\omega). \end{aligned}$$

This is the sum formula for sine. The difference formula for sine is derived as follows:

$$\begin{aligned} \sin(\theta - \omega) &= \sin(\theta + (-\omega)) \\ &= \sin(\theta) \cos(-\omega) + \cos(\theta) \sin(-\omega) \\ &= \sin(\theta) \cos(\omega) - \cos(\theta) \sin(\omega) \end{aligned}$$

To summarize, *the sum and difference formulas for sine and cosine are*

$$\begin{aligned} \cos(\theta + \omega) &= \cos(\theta) \cos(\omega) - \sin(\theta) \sin(\omega) \\ \cos(\theta - \omega) &= \cos(\theta) \cos(\omega) + \sin(\theta) \sin(\omega) \\ \sin(\theta + \omega) &= \sin(\theta) \cos(\omega) + \cos(\theta) \sin(\omega) \\ \sin(\theta - \omega) &= \sin(\theta) \cos(\omega) - \cos(\theta) \sin(\omega) \end{aligned}$$



CHAPTER 15. SUM AND DIFFERENCE FORMULAS

Question 0.14. Find the exact values of the six trigonometric functions for the angle $\theta = \frac{\pi}{12}$

Solution: Observe that

$$\theta = \frac{\pi}{3} - \frac{\pi}{4}.$$

Applying the difference formula for cosine yields

$$\begin{aligned}\cos(\theta) &= \cos\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \cos\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{3}\right)\sin(\pi/4) \\ &= \frac{1}{2} \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{2} + \sqrt{6}}{4}.\end{aligned}$$

Now using the difference formula for sine yields

$$\begin{aligned}\sin(\theta) &= \sin\left(\frac{\pi}{3} - \frac{\pi}{4}\right) \\ &= \sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{4}\right) - \cos\left(\frac{\pi}{3}\right)\sin(\pi/4) \\ &= \frac{\sqrt{3}}{2} \frac{\sqrt{2}}{2} - \frac{1}{2} \frac{\sqrt{2}}{2} \\ &= \frac{\sqrt{6} - \sqrt{2}}{4}.\end{aligned}$$

The other four values follow from the definitions:

$$\begin{aligned}\tan(\theta) &= \frac{\sin(\theta)}{\cos(\theta)} = \frac{\sqrt{6} - \sqrt{2}}{\sqrt{2} + \sqrt{6}} \\ \sec(\theta) &= \frac{1}{\cos(\theta)} = \frac{4}{\sqrt{2} + \sqrt{6}} \\ \cot(\theta) &= \frac{1}{\tan(\theta)} = \frac{\sqrt{6} + \sqrt{2}}{\sqrt{2} - \sqrt{6}} \\ \csc(\theta) &= \frac{1}{\sin(\theta)} = \frac{4}{\sqrt{6} - \sqrt{2}}\end{aligned}$$

Question 0.15. Derive identities for $\tan(\theta \pm \omega)$ in terms of $\tan(\theta)$ and $\tan(\omega)$.

Solution: Calculate as follows:

$$\begin{aligned}
 \tan(\theta + \omega) &= \frac{\sin(\theta + \omega)}{\cos(\theta + \omega)} \\
 &= \frac{\sin(\theta) \cos(\omega) + \cos(\theta) \sin(\omega)}{\cos(\theta) \cos(\omega) - \sin(\theta) \sin(\omega)} \\
 &= \frac{\sin(\theta) \cos(\omega) + \cos(\theta) \sin(\omega)}{\cos(\theta) \cos(\omega) - \sin(\theta) \sin(\omega)} \cdot \frac{\frac{1}{\cos(\theta) \cos(\omega)}}{\frac{1}{\cos(\theta) \cos(\omega)}} \\
 &= \frac{\frac{\sin(\theta)}{\cos(\theta)} + \frac{\sin(\omega)}{\cos(\omega)}}{1 - \frac{\sin(\theta) \sin(\omega)}{\cos(\theta) \cos(\omega)}} \\
 &= \frac{\tan(\theta) + \tan(\omega)}{1 - \tan(\theta) \tan(\omega)}
 \end{aligned}$$

In a similar fashion,

$$\tan(\theta - \omega) = \frac{\tan(\theta) - \tan(\omega)}{1 + \tan(\theta) \tan(\omega)}.$$

!

CHAPTER 16

Double Angle Formulas

The double angle formulas are some simple consequences of the sum and difference formulas. To begin,

$$\cos(2\theta) = \cos(\theta + \theta).$$

The sum formula for cosine then implies,

$$\begin{aligned}\cos(\theta + \theta) &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) \\ &= \cos^2(\theta) - \sin^2(\theta).\end{aligned}$$

Hence, the double angle formula for cosine,

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta).$$

Two equivalent versions of this last identity are also important. First, observe that since $\cos^2(\theta) + \sin^2(\theta) = 1$, $\cos^2(\theta) = 1 - \sin^2(\theta)$. So,

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= (1 - \sin^2(\theta)) - \sin^2(\theta) \\ &= 1 - 2\sin^2(\theta).\end{aligned}$$

This is the second version of the double angle formula. Next, we may also solve for $\sin^2(\theta)$ in the Pythagorean theorem to get $\sin^2(\theta) = 1 - \cos^2(\theta)$. Substituting this into the first version of the double angle formula for cosine yields,

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= \cos^2(\theta) - (1 - \cos^2(\theta)) \\ &= 2\cos^2(\theta) - 1.\end{aligned}$$

To summarize, here are the three *double angle formulas for cosine*

$$\begin{aligned}\cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta) \\ &= 2\cos^2(\theta) - 1 \\ &= 1 - 2\sin^2(\theta).\end{aligned}$$



There is just one double angle formula for sine and it follows from the sum formula for sine.

$$\begin{aligned}\sin(2\theta) &= \sin(\theta + \theta) \\ &= \sin(\theta) \cos(\theta) + \cos(\theta) \sin(\theta) \\ &= 2 \sin(\theta) \cos(\theta).\end{aligned}$$

In other words, *the double angle formula for sine* is

$$\sin(2\theta) = 2 \sin(\theta) \cos(\theta)$$

Question 0.16. Calculate the exact values of the six trigonometric functions at the angle $\pi/8$.

CHAPTER 16. DOUBLE ANGLE FORMULAS

Solution: Start with the observation that $\pi/4 = 2 \cdot (\pi/8)$ and that the angle $\pi/8$ is in the first quadrant. Apply the double angle formula with $\theta = \pi/8$:

$$\begin{aligned}\cos(2\theta) &= 2\cos^2(\theta) - 1 \\ \cos(2 \cdot \pi/8) &= 2\cos^2(\pi/8) - 1 \\ \cos(\pi/4) &= 2\cos^2(\pi/8) - 1 \\ \frac{1}{\sqrt{2}} &= 2\cos^2(\pi/8) - 1\end{aligned}$$

Now solve for $\cos^2(\pi/8)$:

$$\begin{aligned}2\cos^2(\pi/8) &= \frac{1}{\sqrt{2}} + 1 \\ \cos^2(\pi/8) &= \frac{1}{2} \left(\frac{1}{\sqrt{2}} + 1 \right) \\ \cos^2(\pi/8) &= \frac{1}{2} \left(\frac{\sqrt{2}}{2} + 1 \right) \\ \cos^2(\pi/8) &= \frac{1}{2} \frac{\sqrt{2} + 2}{2} \\ \cos^2(\pi/8) &= \frac{\sqrt{2} + 2}{4}\end{aligned}$$

Since $\pi/8$ is in the first quadrant,

$$\cos(\pi/8) = +\sqrt{\frac{\sqrt{2} + 2}{4}} = \frac{\sqrt{\sqrt{2} + 2}}{2}$$

A similar computation for $\sin(\pi/8)$ begins with the double angle formula

$$\cos(2\theta) = 1 - 2\sin^2(\theta).$$

Substituting $\theta = \pi/8$ and solving for $\sin^2(\pi/8)$ yields

$$\sin^2(\pi/8) = \frac{2 - \sqrt{2}}{4}.$$

Since $\pi/8$ is in the first quadrant,

$$\sin(\pi/8) = \frac{\sqrt{2 - \sqrt{2}}}{2}.$$

The value of tangent now follows

$$\tan(\pi/8) = \frac{\sin(\pi/8)}{\cos(\pi/8)} = \frac{\sqrt{2 - \sqrt{2}}}{\sqrt{2 + \sqrt{2}}}$$

The values of the other three trigonometric functions can be computed from this by taking reciprocals.
